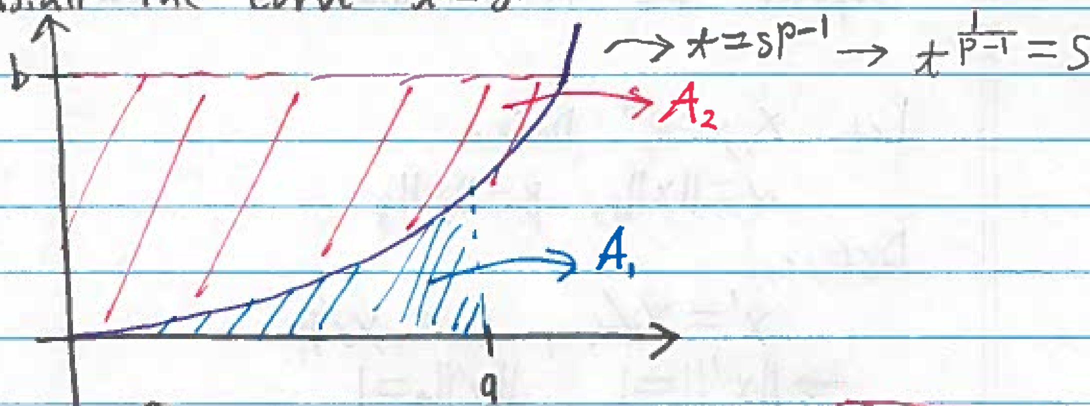
Lecture 4: Infinite Dimensional Spaces

Young's Inequality - If $p \geq 1$ and $\frac{1}{p}+\frac{1}{q}=1$ and $a, b > 0$ then

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q$$

~~proof~~:

Consider the curve $t = s^{p-1}$



$$A_1 = \int_0^a s^{p-1}\,ds = \frac{a^p}{p},$$

$$A_2 = \int_0^b t^{\frac{1}{p-1}}\,dt = \int_0^b t^{q-1}\,dt$$

$$= \frac{b^q}{q}.$$

$\frac{1}{p}+\frac{1}{q}=1$

$\Rightarrow q = \frac{p}{p-1}$

$\Rightarrow q-1 = \frac{1}{p-1}.$

$$\Rightarrow ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q$$

Area of rectangle $\quad A_1 \quad A_2$

Minkowski's Inequality - If $p \geq 1$ then for $x, y \in \mathbb{R}^n$:

$$\|x+y\|_p \leq \|x\|_p + \|y\|_p.$$

~~proof~~:

$$|x_i+y_i|^p \leq \big||x_i|+|y_i|\big|^p$$
$$\leq \big||x_i|+|y_i|\big| \cdot \big||x_i|+|y_i|\big|^{p-1}$$
$$\leq |x_i| \cdot \big||x_i|+|y_i|\big|^{p-1} + |y_i| \cdot \big||x_i|+|y_i|\big|^{p-1}$$

$$\Rightarrow \sum_{i=1}^{n} |x_i+y_i|^p \leq \sum_{i=1}^{n} |x_i| \cdot \big||x_i|+|y_i|\big|^{p-1} + \sum_{i=1}^{n} |y_i| \cdot \big||x_i|+|y_i|\big|^{p-1}$$

$$\Rightarrow \sum_{i=1}^{n} |x_i+y_i|^p \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} (|x_i|+|y_i|)^p\right)^{1/q}$$
$$+ \left(\sum_{i=1}^{n} |y_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} (|x_i|+|y_i|)^p\right)^{1/q}$$

$$\Rightarrow \sum_{i=1}^{n} |x_i+y_i|^p \leq \left(\sum_{i=1}^{n} (|x_i|+|y_i|)^p\right)^{1/q} \left(\|x\|_p + \|y\|_p\right)$$

$$\Rightarrow \|x+y\|_p \leq \|x\|_p + \|y\|_p.$$

Sequence Spaces:

- $\ell^p(\mathbb{N}, \mathbb{R}) = \begin{cases} \{x_n \in \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty\} & (1 \leq p < \infty). \\ \{x_n \in \mathbb{R} : \sup_n |x_n| < \infty\} & (p = \infty) \end{cases}$

- Let $x \in \ell^p$, then $\|x\|_p = \begin{cases} \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p} & \text{if } 1 \leq p < \infty \\ \sup_n |x_n| & \text{if } p = \infty \end{cases}$

- If $x \in \ell^p$, and $\alpha \in \mathbb{R}$ then $\|\alpha x\|_p = |\alpha| \cdot \|x\|_p$

Definition - A sequence $x^{(n)} \in \ell^2$ is componentwise convergent if for each fixed K the sequence of real numbers $x_K^{(n)}$ is convergent.

* If $x^{(n)} \to x$ in ℓ^2 then $x^{(n)}$ converges componentwise
* If $x^{(n)} \to x$ componentwise then $x^{(n)}$ may not converge to x in ℓ^2.

example:

$$x^{(1)} = (1, 0, 0, \ldots)$$
$$x^{(2)} = (0, 1, 0, \ldots)$$
$$x^{(3)} = (0, 0, 1, \ldots)$$
$$\vdots$$
$$x \to (0, 0, 0, \ldots) \quad \text{componentwise}$$

* Componentwise convergence is weaker than ℓ^2 convergence.

Theorem - If a sequence $x^{(n)}$ is bounded, then $x^{(n)}$ has a subsequence that is componentwise convergent.

proof:

Let $x^{(n)}$ be a bounded sequence in ℓ^2.

1. The sequence $x_1^{(n)}$ is bounded in $\mathbb{R}$ and therefore has a convergent subsequence. $x_1^{(n_K)}$ with limit x_1.
2. The sequence $x_2^{(n_K)}$ is bounded in $\mathbb{R}$ and therefore has a convergent subsequence $x_2^{(n_{K_\ell})}$ with limit x_2
3. Continue recursively get a subsequence $x^{n_{K_\ell}}$ such that first k-components converge.