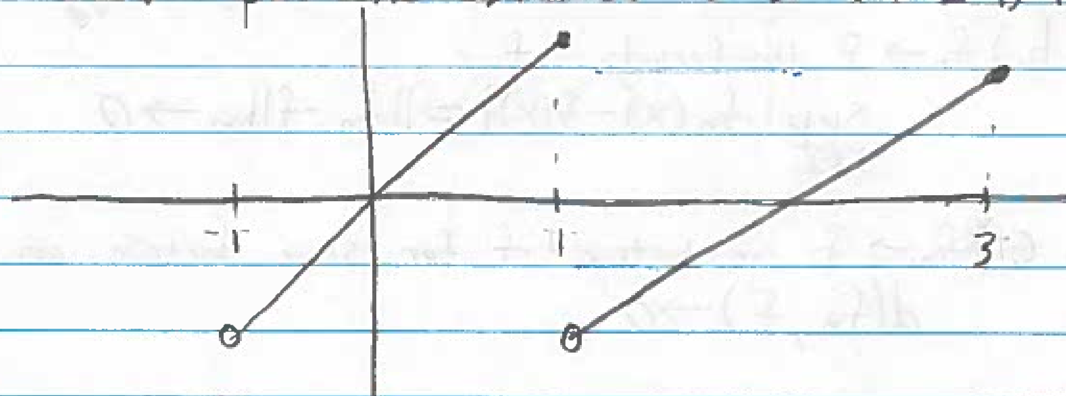
Read:
RR - 1.5.1
HN - 2.1, 2.2,

Lecture 5: Convergence of Functions

example:

$f(x) =$ periodic extension of x on $[-1, 1]$.



$$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

$$\Rightarrow \int_{-1}^{1} f(x)\sin(m\pi x) = \int_{-1}^{1} b_m \sin^2(m\pi x)\,dx$$

$$\Rightarrow \int_{-1}^{1} x\sin(m\pi x)\,dx = b_m$$

$$\Rightarrow \left.\frac{-x\cos(m\pi x)}{m\pi}\right|_{-1}^{1} = b_m$$

$$\Rightarrow b_m = \frac{2}{m\pi}(-1)^{m+1}$$

Therefore,

$$f(x) \text{ "}=\text{" } \sum_{n=1}^{\infty} \frac{2}{n\pi}(-1)^{n+1} \sin(n\pi x)$$

This gives us a sequence of functions

$$f_N(x) = \sum_{n=1}^{N} \frac{2}{n\pi}(-1)^{n+1} \sin(n\pi x).$$

Recall:

Let $f_n : \mathbb{X} \mapsto \mathbb{R}$, where $\mathbb{X}$ is a metric space.

a.) $f_n \to f$ pointwise: if for all for all $x \in \mathbb{X}$
$f_n(x) \to f(x)$.

b.) $f_n \to f$ uniformly if
$$\sup_{x\in\mathbb{X}} |f_n(x) - f(x)| = \|f_n - f\|_\infty \to 0$$

c.) $f_n \to f$ in metric if for some metric on functions
$$d(f_n, f) \to 0$$

Return to example:
$$f_N(x) = \sum_{n=1}^{N} \frac{2}{n}(-1)^{n+1} \sin(n\pi x)$$

a.) f_N does not converge pointwise to x:
$$f_N(\pi) = 0 \Rightarrow \lim_{N\to\infty} f_N(\pi) = 0 \neq \pi$$

b.) f_N does not converge uniformly:
Gibb's phenomenon:
*Very rapid oscillations near endpoints
(see next page)

c.) Define $d(f,g)$ by:
$$d(f,g) = \left(\int_{-1}^{1} |f-g|^2 \, dx\right)^{1/2}$$

$$\Rightarrow d(f_N, f) = \left(\int_{-1}^{1} \left|\sum_{n=1}^{N} \frac{2}{n}(-1)^{n+1} \sin(n\pi x) - x\right|^2 dx\right)^{1/2}$$

$$= \|f_N - x\|_{L^2}$$

mean squared norm.

Now,

$$\|f_N - x\|_{L^2}^2 = \|f_N\|_{L^2}^2 - 2\int_{-1}^{1} \frac{2}{\pi n}(-1)^{n+1}\sin(n\pi x)\,x\,dx + \|x\|_{L^2}^2$$

However,

$$\|f_N\|_{L^2}^2 = \int_{-1}^{1} \sum_{n\geq 1}^{N} \frac{4}{\pi^2 n^2}\sin^2(m\pi x)\,dx - \frac{8}{\pi^2 n^2} + \|x\|_{L^2}^2$$

$$= \|\underbrace{x}_{L^2}\|_{L^2}^2 - \underbrace{\frac{4}{\pi^2 n^2}}_{\ell^2}$$

We need

$$\left\|\frac{4}{\pi^2 n^2}\right\|_{\ell^2} \longrightarrow \|x\|_{L^2}$$ (This is true but it will take awhile to show)

The upshot is that $f_N \to f$ in L^2.

<u>Theorem</u>- Let f_n be a sequence of bounded continuous, real-valued functions on a metric space (X, d). If $f_n \to f$ uniformly, then f is continuous.

proof:

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)|$$

Since $f_n \to f$ uniformly there exists $N \in \mathbb{N}$ such that $n \geq N$ implies

$$|f(x) - f_n(x)| < \varepsilon/3 \quad \text{and} \quad |f_n(y) - f(y)| < \varepsilon/3$$

Since f_n is continuous at x there exists $\delta > 0$ such that $d(x,y) < \delta$ implies $|f_n(x) - f_n(y)| < \varepsilon/3$. Therefore, $d(x,y) < \delta$ implies

$$|f(x) - f(y)| < \varepsilon.$$