Lecture 6: Continuous Functions and Lebesgue Spaces

Theorem - Let f_n be a sequence of bounded, continuous, real valued functions on a metric space (X, d). If $f_n \to f$ uniformly then f is continuous.

proof:

By the triangle inequality:

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)|$$

Since $f_n \to f$ uniformly, there is an $N \in \mathbb{N}$ such that $n \geq N$ implies

$$|f(x) - f_n(x)| < \varepsilon/3, \quad |f(y) - f_n(y)| < \varepsilon/3.$$

Since f_n is continuous there exists $\delta > 0$ such that $d(x,y) < \delta \Rightarrow |f_n(x) - f_n(y)| < \varepsilon/3$.

$\blacksquare$

Definition - A Banach space is a complete normed linear space.

Definition - Let X be a metric space. The space of continuous real-valued functions $f: X \to \mathbb{R}$ is denoted by $C(X)$.

Theorem - Let $K \subset X$ be compact. The space $(C(K), \|\cdot\|_\infty)$ is a Banach space.

proof:

Let $f_n \in C(X)$ be Cauchy with respect to $\|\cdot\|_\infty$.

Therefore, $\forall \varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $m, n \geq N$

$$\Rightarrow \sup_{x \in X} |f_n(x) - f_m(x)| = \|f_n - f_m\|_\infty < \varepsilon.$$

Therefore, for each x, $f_n(x)$ is Cauchy in $\mathbb{R}$ and thus

$$f_n(x) \to f(x).$$

Finally, for all $x \in X$, $n \geq N$ implies

$$\begin{aligned} |f_n(x) - f(x)| &\leq |f_n(x) - f_m(x)| + |f_m(x) - f(x)| \\ &\leq \|f_n - f_m\|_\infty + |f_m(x) - f(x)| \\ &\leq \varepsilon + |f_m(x) - f(x)| \end{aligned}$$

$$\Rightarrow |f_n(x) - f(x)| \leq \varepsilon + \lim_{m\to\infty} |f_m(x) - f(x)|$$

$$\Rightarrow |f_n(x) - f(x)| \leq \varepsilon.$$

Consequently, $f_n \to f$ in $\|\cdot\|_\infty$. f is continuous by the above theorem.

Examples:

1. $K = \{x_1, \ldots, x_n\}$ is a finite collection with metric

$$d(x_i, x_j) = \begin{cases} 1 & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases}$$

A function $f: K \to \mathbb{R}$ can be identified with a point $(y_1, \ldots, y_n) \in \mathbb{R}^n$ by $f(x_i) = y_i$, and

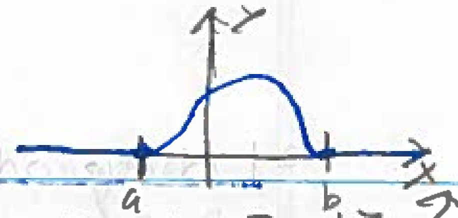
$$\|f\|_\infty = \max_{1 \leq i \leq n} |y_i|$$

Therefore, $C(K) \sim (\mathbb{R}^n, \|\cdot\|_\infty)$ (Finite Dimensional)

* Functions are defined by their outputs.

2. $K = [0,1]$. $C(K)$ is infinite dimensional
3. $C_b(X) = \{f \in C(X) : \|f\|_\infty < \infty\}$ is a Banach space.
4. Definition - The support, supp f, of a function $f: X \to \mathbb{R}$ on X is defined by:

$\mathrm{supp}\, f = \overline{\{x \in X ; f(x) \neq 0\}}$

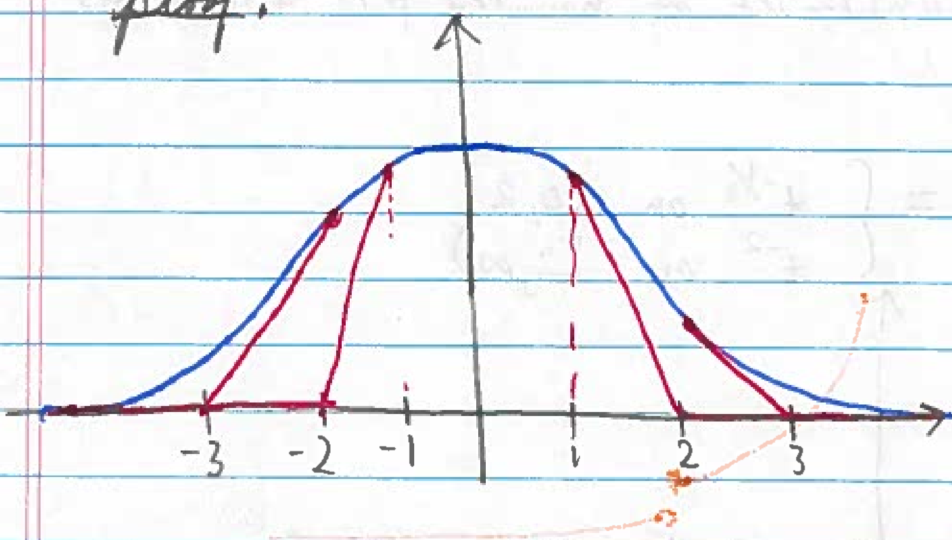


$\mathrm{supp}\, f = [a, b]$

The space of functions with compact support is denoted by $C_c(X)$.

* $C_c(X)$ is not complete with respect to $\|\cdot\|_\infty$.

proof:



$$f_n(x) = \begin{cases} e^{-x^2}, & |x| \leq n \\ -e^{-n^2}(x-n) + e^{-n^2}, & 0 \leq |x| \leq n+1 \\ 0, & |x| > n+1 \end{cases}$$

$\Rightarrow \|e^{-x^2} - f_n\|_\infty \leq 2e^{-n^2} \to 0$

However, $\mathrm{supp}(f) = (-\infty, \infty)$.

5. The closure of $C_c(X)$ with respect to $\|\cdot\|_\infty$ is $C_0(X)$ and is a Banach space.

Space of functions that vanish at ∞.

Lebesgue Spaces

Definition - For any a, b, $(-\infty \le a < b \le \infty)$ and $1 \le p < \infty$ define

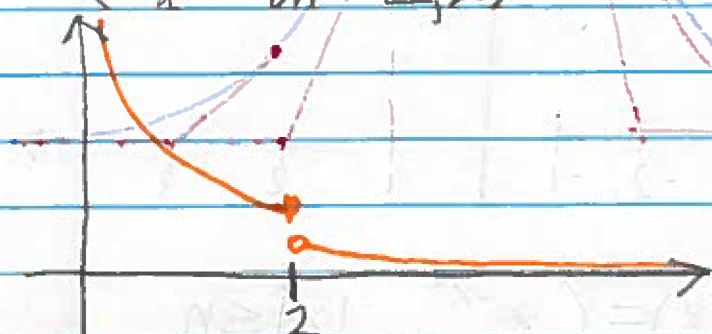
$L_0^p([a,b])$ = set of piecewise continuous functions for which $\int_a^b |f(t)|^p dt < \infty$.

Define,

$L_0^\infty([a,b])$ = set of bounded p.c. functions.

example:

$$g(t) = \begin{cases} t^{-1/3} & \text{on } [0,2] \\ t^{-2} & \text{on } [2,\infty) \end{cases}$$



- $\int_0^2 |g(t)|^p dt = \int_0^2 t^{-p/3} dt = \begin{cases} \infty & \text{if } p \ge 3 \\ < \infty & \text{if } p < 3 \end{cases}$
- $\int_2^\infty |g(t)|^p dt = \int_2^\infty t^{-2p} dt < \infty$

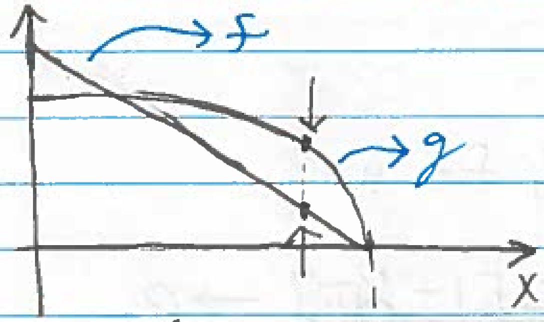
$\Rightarrow g \in L_0^p([a,b])$ if $1 \le p < 3$.

Definition - The L^p metric on $L_0^p([a,b])$ is given by

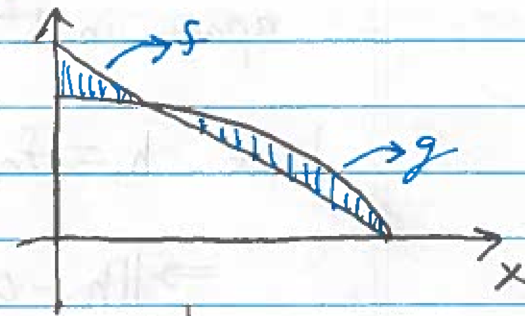
$\|f-g\|_{L^p} = \left(\int_a^b |f-g|^p dx\right)^{1/p}$ for $1 \le p < \infty$

$\|f-g\|_{L^\infty} = \sup_{x \in (a,b)} |f-g|$ for $p = \infty$.

What do the L^p norms measure?



L_0^∞ - measures biggest gap



L_0^1 - measures area between curves

L_0^2 - 1. Square $|f(x)-g(x)|$ (expand or contract area)
2. Find total distorted area
3. Take square root.

(r.m.s.) error

Hölder's and Minkowski:

1. $\int_a^b |f(x)g(x)| \le \|f\|_p \cdot \|g\|_q \quad \left(\frac{1}{p}+\frac{1}{q}=1\right)$.
2. $\|f+g\|_p \le \|f\|_p + \|g\|_p$.

Example:

Consider the sequence of functions defined by

$$f_n(x) = \begin{cases} 0 & \text{if } 0 \le x \le 1/n \\ 1/\sqrt{x} & \text{if } 1/n \le x \le 1. \end{cases}$$

Let $g(x) = 0$.

$$\|f_n - g\|_1 = \int_{1/n}^1 \frac{1}{\sqrt{x}}\,dx = 2\cdot\left[1 - 1/\sqrt{n}\right].$$

$$\|f_n - g\|_2 = \left(\int_{1/n}^1 \frac{1}{x}\,dx\right)^{1/2} = \sqrt{\ln(n)}$$

* Functions stay finitely apart in L^1 but infinitely apart in L^2.

Let $h_n = f_n/\sqrt{\ln(n)}$, for $n \geq 1$.

$$\Rightarrow \|h_n - 0\|_1 = \frac{2[1 - 1/\sqrt{n}]}{\sqrt{\ln(n)}} \to 0$$

$$\|h_n - 0\|_2 = 1$$

* Impossible for L_0^p metrics to be equivalent.
* L_0^2 unit ball is not compact!!

Proposition - If $f_n \to f$ in $L_0^2([0,1])$ then $f_n \to f$ in $L_0^1([0,1]$

proof:
$$\int_0^1 |f_n(t) - f(t)|\,dt \leq \left(\int_0^1 |f_n(t) - f(t)|^2 dt\right)^{1/2} \left(\int_0^1 1^2 dt\right)^{1/2}$$
$$= \|f_n - f\|_{L_0^2} \to 0.$$

Example:

On $(0, \infty)$ define
$$f_n(t) = \begin{cases} 1/n, & \text{if } n < t \leq 2n \\ 0, & \text{o.w.} \end{cases}$$

$$\Rightarrow \int_0^\infty |f_n(t)|\,dt = 1$$

$$\int_0^\infty |f_n(t)|^2 dt = 1/n \to 0.$$