# Lecture 7: Arzela-Ascoli

Lemma - If $f_n \subset \mathcal{C}([0,1])$ is bounded, then $f_n$ has a subsequence $f_{n_k}$ such that for all $r \in \mathbb{Q}$ $f_{n_k}$ is pointwise convergent.

~~proof~~:

Since $\mathbb{Q}$ is countable it follows that there exists a $r_i \in \mathbb{Q}$ satisfying $r_i \neq r_j$ for $i \neq j$ and $\mathbb{Q} = \bigcup_{i=1}^{\infty} \{r_i\}$.

1. For $i=1$, the sequence of values $f_n(r_1)$ has a convergent subsequence $f_{n_{k}}(r_1) \to f(r_1)$.

2. For $i=2$, the sequence of values $f_{n_{k_1}}(r_2)$ has a convergent subsequence $f_{n_{k_2}}(r_2) \to f(r_2)$

3. Continue recursively you get nested subsequences $f_{n_{k_1}}, f_{n_{k_2}}, \ldots$ each satisfying
$$f_{n_{k_1}}(r_1) \to f(r_1)$$
$$f_{n_{k_2}}(r_1) \to f(r_1) \text{ and } f_{n_{k_2}}(r_2) \to f(r_2)$$
$$\vdots$$

4. Define a subsequence $f_{n_k}$ by:
$$f_{n_1} = f_{n_{1_1}}$$
$$f_{n_2} = f_{n_{2_2}}$$
$$f_{n_3} = f_{n_{3_3}}$$
$$\vdots$$

5. It follows that for all $r \in \mathbb{Q}$:
$$\lim_{k \to \infty} f_{n_k}(r) = f(r).$$

*We want to improve the previous theorem to full compactness.

**Definition** - A set $K \subset \mathcal{C}([0,1])$ is uniformly equicontinuous if given any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $f \in K$ if $|x-y| \leq \delta$ then $|f(x) - f(y)| \leq \varepsilon$.

**Theorem** - If $f_n \subset \mathcal{C}([0,1])$ is pointwise convergent on $\mathbb{Q} \cap [0,1]$, and $f_n$ is uniformly equicontinuous, then $f_n$ is uniformly Cauchy.

~~prof~~:

Let $\varepsilon > 0$.

1. By uniform equicontinuity there exists $\delta > 0$ such that if $|x_2 - x_1| < \delta$ then $|f_n(x_2) - f_n(x_1)| < \varepsilon/3$.

2. Choose rational numbers $\{r_1, \ldots, r_j\}$ such that for any $x \in [0,1]$ there exists $r_i$ such that $|x - r_i| < \delta$.

3. For each $r_i$, the sequence $f_n(r_i)$ is Cauchy so there exists $N_i \in \mathbb{N}$ such that $m,n \geq N_i \Rightarrow |f_n(r_i) - f_m(r_i)| < \varepsilon/3$

4. Let $N = \max\{N_1, \ldots, N_j\}$. For any $x \in [0,1]$ select $r_i$ such that $|x - r_i| < \delta$. Therefore,

$$\begin{aligned}|f_n(x) - f_m(x)| &\leq |f_n(x) - f_n(r_i)| + |f_n(r_i) - f_m(r_i)| \\ &\quad + |f_m(r_i) - f_m(x)| \\ &\leq \varepsilon. \end{aligned}$$

■

Theorem - If $K \subset \mathcal{C}([0,1])$ is closed, bounded, and uniformly equicontinuous, then $K$ is compact.

proof:

Let $f_n \in K$. Then there exists a subsequence $f_{n_k}$ that converges on $\mathbb{Q} \cap [0,1]$. Consequently, $f_{n_k}$ is uniformly Cauchy and thus by completeness of $K$ converges. ■

* How to guarantee equicontinuity?

Lemma - If $K \subset \mathcal{C}([0,1])$ is a set of differentiable functions, and if there exists $M>0$ such that $|f'(x)| \le M$ for $x \in [0,1]$ and all $f \in K$, then $K$ is uniformly equicontinuous

proof:

By the Fundamental Theorem of Calculus:

$$|f(y)-f(x)| = \left|\int_x^y f'(s)\,ds\right|$$

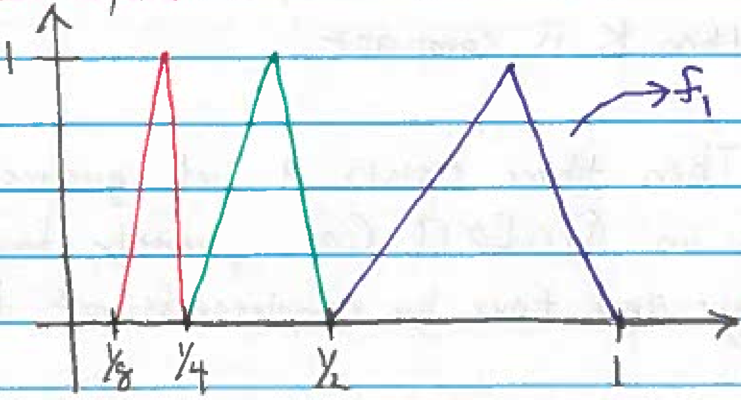
$$\le \int_x^y |f'(s)|\,ds$$

$$\le \int_x^y M\,ds$$

$$\le M \cdot |y-x|.$$

Therefore, if $\delta < \varepsilon/M$ then $|y-x|<\delta \Rightarrow |f(y)-f(x)|<\varepsilon$ ■

**Example:**




$K = \{f_n : n \in \mathbb{N}\}$

$\|f_n\|_\infty = 1$, $\|f_n - f_m\| = 1$ for $m \neq n$.

$\Rightarrow$ No convergent subsequences.

* closed, bounded subset of $C([0,1])$ which is not compact.

* $\mathcal{F}$ is not equicontinuous because the slopes get steeper.

**Definition:** A function $f: X \to \mathbb{R}$ is Lipshitz continuous on $X$ if there exists a constant $C \geq 0$ such that
$$|f(x) - f(y)| \leq C\, d(x,y), \text{ for all } x, y \in X.$$

**Example:**

1. $x^3$ is Lipshitz on $[0,1]$ but not $\mathbb{R}$.

*proof*

(For $0 \leq x \leq 1$): $|x^3 - y^3| \leq 3|x - y|$ (Mean value theorem).

2. $x^{1/3}$ is not Lipshitz on $[0,1]$.

*proof*
$$\lim_{x \to 0^+} \frac{x^{1/3}}{x} = \lim_{x \to 0^+} \frac{|x^{1/3} - 0|}{|x - 0|} = \infty.$$

3. $|x|$ is Lipschitz because

$$|f(x)-f(y)| = \big||x|-|y|\big| \leq |x-y|.$$

Definition - $\mathrm{Lip}(f) = \sup_{x \neq y} \frac{|f(x)-f(y)|}{|x-y|}$. Let $K \subseteq X$ be compact. Define

$$\mathcal{F}_M = \{ f \in K : \mathrm{lip}(f) \leq M \}$$

* $\mathcal{F}_M$ is equicontinuous since if $\varepsilon > 0$, and $\delta = \varepsilon/M$, then $d(x,y) < \delta \Rightarrow |f(x)-f(y)| \leq \varepsilon$.