Lemma: If $f_n \in C([0,1])$ is bounded, then $f_n$ has a subsequence $f_{n_k}$ such that for all $r \in \mathbb{Q}$, $f_{n_k} (r)$ is pointwise convergent.

Proof:
Since $\mathbb{Q}$ is countable it follows that there exists a $r_i, i \in \mathbb{Q}$ satisfying $r_i \neq r_j$ for $i \neq j$ and $Q = \bigcup_{i} \mathbb{Q}$.

1. For $i = 1$, the sequence of values $f_n (r_1)$ has a convergent subsequence $f_{n_k} (r_1) \to f (r_1)$.

2. For $i = 2$, the sequence of values $f_{n_k} (r_2)$ has a convergent subsequence $f_{n_{k_2}} (r_2) \to f (r_2)$.

3. Continue recursively you get nested subsequences $f_{n_{k_1}}, f_{n_{k_2}}, \ldots$ each satisfying
   - $f_{n_{k_1}} (r_1) \to f (r_1)$
   - $f_{n_{k_2}} (r_1) \to f (r_1)$ and $f_{n_{k_2}} (r_2) \to f (r_2)$
   - $\vdots$

4. Define a subsequence $f_{n_k}$ by:
   - $f_{n_1} = f_{n_1}$
   - $f_{n_2} = f_{n_{k_2}}$
   - $f_{n_3} = f_{n_{k_3}}$
   - $\vdots$

5. It follows that for all $r \in \mathbb{Q}$:
   $$ \lim_{k \to \infty} f_{n_k} (r) = f (r). $$
We want to improve the previous theorem to full compactness.

**Definition** - A set $K \subseteq C([0,1])$ is uniformly equicontinuous if given any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $f \in K$ if $|x - y| \leq \delta$ then $|f(x) - f(y)| \leq \varepsilon$.

**Theorem** - If $f_n \subseteq C([0,1])$ is pointwise convergent on $Q \subseteq [0,1]$, and $f_n$ is uniformly equicontinuous, then $f_n$ is uniformly Cauchy.

**Proof**

Let $\varepsilon > 0$.

1. By uniform equicontinuity there exists $\delta > 0$ such that if $|x_2 - x_1| \leq \delta$ then $|f_n(x_2) - f_n(x_1)| < \frac{\varepsilon}{3}$.

2. Choose rational numbers $r_1, \ldots, r_j$ such that for any $x \in [0,1]$ there exists $r_i$ such that $|x - r_i| < \delta$.

3. For each $r_i$, the sequence $f_n(r_i)$ is Cauchy so there exists $N_i \in \mathbb{N}$ such that $m, n > N_i \Rightarrow |f_n(r_i) - f_m(r_i)| < \frac{\varepsilon}{3}$.

4. Let $N = \max N_1, \ldots, N_j$. For any $x \in [0,1]$ select $r_i$ such that $|x - r_i| < \delta$. Therefore,

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f_n(r_i)| + |f_n(r_i) - f_m(r_i)| + |f_m(r_i) - f_m(x)|$$

$$\leq \frac{\varepsilon}{3}.$$
Theorem - If \( K \subseteq C([0,1]) \) is closed, bounded, and uniformly equicontinuous, then \( K \) is compact.

Proof:
Let \( f_n \in K \). Then there exists a subsequence \( f_{n_k} \) that converges on \( \mathbb{R} \) \( \cap \] [0,1]. Consequently, \( f_{n_k} \) is uniformly Cauchy and thus by completeness of \( K \) converges.

*How to guarantee equicontinuity?*

Lemma - If \( K \subseteq C([0,1]) \) is a set of differentiable functions, and if there exists \( M > 0 \) such that \( |f'(x)| \leq M \) for \( x \in [0,1] \) and all \( f \in K \), then \( K \) is uniformly equicontinuous.

Proof:
By the Fundamental Theorem of Calculus:
\[
|f(y) - f(x)| = \left| \int_x^y f'(s) \, ds \right|
\]
\[
\leq \int_x^y |f'(s)| \, ds
\]
\[
\leq \int_x^y M \, ds
\]
\[
= M \cdot |y - x|
\]

Therefore, if \( \delta = \frac{\varepsilon}{M} \) then \( |y - x| < \delta \Rightarrow |f(y) - f(x)| < \varepsilon \).
**Example:**

Let $k = \{ f_n : n \in \mathbb{N} \}$

If $\|f_n\|_\infty = 1$, $\|f_n - f_m\|_1 = 1$ for $m \neq n$.

$\Rightarrow$ No convergent subsequences.

* * closed, bounded subset of $C([0,1])$ which is not compact.

* $k$ is not equicontinuous because the slopes get steeper.

**Definition:** A function $f : X \rightarrow \mathbb{R}$ is Lipschitz continuous on $X$ if there exists a constant $C \geq 0$ such that

$$|f(x) - f(y)| \leq C \cdot d(x, y) \quad \text{for all} \quad x, y \in X.$$

**Example:**

1. $x^3$ is Lipshitz on $[0,1]$ but not $\mathbb{R}$.

   **Proof:**

   (For $x \neq 0$), $|x^3 - y^3| \leq 3|x - y|$ (Mean Value Theorem).

2. $x^{1/3}$ is not Lipshitz on $[0,1]$.

   **Proof:**

   $$\lim_{x \to 0^+} \frac{x^{1/3} - 0}{x - 0} = \lim_{x \to 0^+} \frac{1}{3x^{2/3}} = \infty.$$
3. \( |x| \) is Lipschitz because
\[
|f(x) - f(y)| = |x| - |y| \leq |x - y|.
\]

**Definition.** \( \text{Lip}(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|} \). Let \( K \subseteq \mathbb{R} \) be compact. Define
\[
\mathcal{F}_M = \{ f \in K : \text{Lip}(f) \leq M \}.
\]

* \( \mathcal{F}_M \) is equicontinuous since if \( \varepsilon > 0 \), and \( M = \varepsilon \), then \( d(x, y) < \varepsilon \Rightarrow |f(x) - f(y)| \leq \varepsilon \).