Lecture 8: Ordinary Differential Equations

\[ \frac{du}{dt} = f(t, u) \]

\[ f: \mathbb{R}^2 \rightarrow \mathbb{R} \text{ is continuous.} \]

A solution is a continuously differentiable function \( u: I \rightarrow \mathbb{R} \) such that

\[ \frac{du}{dt} = f(t, u(t)), \]

where \( I \) is an open interval.

\( I = \mathbb{R} \) (local solution)

\( I = \mathbb{R} \) (global solution).

Examples:

1. \( \frac{du}{dt} = au \Rightarrow u(t) = u_0 e^{at} \)

   \( u(0) = u_0 \)

2. \( \frac{du}{dt} = u^3 \Rightarrow \int_{u_0}^u \frac{1}{u^3} du = \int_0^t dt \)

   \( u(0) = u_0 \Rightarrow \frac{1}{2} \left( \frac{1}{u_0^2} - \frac{1}{u^2} \right) = t \)

   \[ \Rightarrow \frac{1}{2} \frac{1}{u^2} - 2t = \frac{1}{u_0^2} - \frac{1}{u^2} \]

   \[ \Rightarrow 1 - 2tu_0^2 = \frac{1}{u_0^2} \]

   \[ \Rightarrow u(t) = \frac{u_0}{\sqrt{1 - 2tu_0^2}} \]

Local Existence until \( t = \frac{1}{2u_0^2} \).
3. \( \frac{dv}{dt} = v^{3/2} \Rightarrow \int_0^v u^{-3/2} du = \int_0^t dt \)

\( u(0) = 0 \Rightarrow \frac{3}{2} (v^{3/2}) = t \)

\( \Rightarrow v^{3/2} = \frac{2}{3} t \)

\( \Rightarrow u(t) = \left( \frac{2}{3} \right)^{2/3} t^{3/2} \)

Not the only solution:

\( u(t) = \begin{cases} 0 & 0 \leq t \leq a - \alpha \ (a > 0), \\ \left( \frac{2}{3} \right)^{2/3} (t-a)^{3/2} & 0 \leq t < a \end{cases} \)

\( \rightarrow \text{Solutions are not unique.} \)

\[ u(t) \]

\( \alpha \rightarrow t \)

\text{Theorem:} Suppose \( f(t, u) \) is a continuous function on \( \mathbb{R}^2 \).

Then for every \( (t_0, u_0) \), there is an open interval \( I \subset \mathbb{R} \) that contains \( t_0 \), and a continuously differentiable function \( u: I \rightarrow \mathbb{R} \) that satisfies

\[ \frac{dv}{dt} = f(t, u) \]

\[ u(t_0) = u_0. \]
proof:

1. Pick $T_1 > 0$ and let 
   
   $I_i = \{ x \in I ; |x-x_i| \leq T_1 \}$ 

   Partition $I_i$ into $2N$ subintervals of length $h$. 
   where $T_i = Nh$, and let 
   
   $t_k = x_0 + kh$ for $-N \leq k \leq N$ 

   ![Graph showing partition into subintervals](image)

2. We construct an approximate solution $u_k(t)$. 
   
   Let $u_k(x_0) = a_k$. Approximate 
   
   \[
   \frac{du_k}{dt} \approx \frac{a_{k+1} - a_k}{h} = f(t_k, a_k).
   \]

   Hence, 
   
   $u_k(t) = a_k + b(t-x_k)$. 

   ![Graph showing piecewise linear function](image)

3. \[
   \left| \frac{du_k}{dt} - f(t, u_k(t)) \right| = \left| f(t_k, a_k) - f(t, a_k + b(t-x_k)) \right| 
   \]
   \[
   \left| x - x_k \right| \leq h, \quad \left| a_k + b(t-x_k) - a_k \right| \leq |bk| h 
   \]
4. We choose an $L$ and $T \leq T_1$ such that for all $\varepsilon$ the graph of $u_\varepsilon$ is contained in $R$ given by

$$R = \mathcal{C}(t, u) : |t - t_0| \leq T, |u - u_0| \leq L.$$  

Let $R_1$ be defined by

$$R_1 = \mathcal{C}(t, u) : |t - t_0| \leq T_1, |u - u_0| \leq L$$

Let $M = \sup \delta f((t, u)) : (t, u) \in R_1, T = \min \{T_1, 5M\}$

5. $R$ is compact which implies $f$ is uniformly continuous.

$$|f(s, u) - f(t, v)| \leq \varepsilon \text{ for } |s - t| \leq \delta \text{ and } |u - v| \leq \varepsilon$$

6. Each $u_\varepsilon$ is Lipschitz with $\text{Lip}(u_\varepsilon) \leq M$.

By Arzelà-Ascoli, $u_\varepsilon \to u$.

7. $u_\varepsilon(t) = u_\varepsilon(t_0) + \int_{t_0}^{t} \frac{du_\varepsilon}{ds} ds$

$$= u_\varepsilon(t_0) + \int_{x_0}^{x_\varepsilon(t)} f(s, u_\varepsilon(s)) ds + \int_{x_0}^{x_\varepsilon(t)} \int_{s}^{x_\varepsilon(t)} du_\varepsilon(s, u) du ds$$

Take limit as $\varepsilon \to 0$, $u(t) = u(t_0) + \int_{t_0}^{t} f(s, u(s)) ds.$
Theorem - Suppose \( \psi(t) \geq 0, \varphi(t) \geq 0 \) are continuously differentiable functions defined on \( 0 \leq t \leq T \) and \( u_0 > 0 \). Then

\[
\varphi(t) \leq u_0 + \int_0^t \psi(s) u(s) \, ds.
\]

then

\[
\varphi(t) = u_0 \exp \left( \int_0^t \psi(s) \, ds \right).
\]

\[\text{Proof:}\]

Let \( u = u_0 + \int_0^t \psi(s) u(s) \, ds \). Then,

\[
\frac{du}{dt} = \psi(t) \cdot u(t) \leq \varphi(t) \cdot u(t)
\]

\[
\Rightarrow \frac{d}{dt}(\ln u) = \frac{\frac{du}{dt}}{u} \leq \varphi(t)
\]

\[
\Rightarrow \ln u(t) = \ln(u_0) + \int_0^t \psi(s) \, ds
\]

\[
\Rightarrow u(t) = u_0 \exp \left( \int_0^t \psi(s) \, ds \right).
\]

Theorem - If \( f \) is a Lipschitz continuous function of \( u \) uniformly in \( t \) then the solution of

\[
\frac{du}{dt} = f(t, u), \quad u(t_0) = u_0.
\]

is unique.

\[\text{Proof:}\]

Suppose \( u, v \) solve \( \psi \). Therefore,

\[
u(t) - v(t) = \int_0^t [f(s, u(s)) - f(s, v(s))] \, ds
\]

\[
\Rightarrow |u(t) - v(t)| \leq \int_0^t |f(s, u(s)) - f(s, v(s))| \, ds = \int_0^t |u(s) - v(s)| \, ds.
\]

Therefore, by Gronwall's inequality

\[
|u(t) - v(t)| \leq |u(t_0) - v(t_0)| \exp \left( \int_0^t |u(s) - v(s)| \, ds \right) = 0.
\]