Quiz 10

Analysis

April 13, 2018

1. Suppose that $k : [a,b] \times [a,b] \mapsto \mathbb{R}$ is a continuous function such that

$$\sup_{a \le x \le b} \left\{ \int_a^b |k(x,y)|\, dy \right\} < 1,$$

and g is a continuous function. Prove that there is a unique continuous $f : [a,b] \mapsto \mathbb{R}$ that satisfies:

$$f(x) = g(x) + \int_a^b k(x,y) f(y)\, dy.$$

Define $T: \mathcal{C}([a,b]) \to \mathcal{C}([a,b])$ by

$$Tf = g(x) + \int_a^b K(x,y) f(y)\, dy.$$

Therefore,

$$\begin{aligned} |(Tf - Th)(x)| &= \left| \int_a^b K(x,y)(f(y) - h(y))\, dy \right| \\ &\le \int_a^b K(x,y) |f(y) - h(y)|\, dy \\ &\le \|f-h\|_\infty \int_a^b K(x,y)\, dy \\ &< \|f-h\|_\infty . \end{aligned}$$

$$\Rightarrow \|Tf - Th\|_\infty < \|f-h\|_\infty .$$

Therefore, since $(\mathcal{C}([a,b]), \|\cdot\|_\infty)$ is complete it follows from the contraction mapping theorem that there exists a unique $f \in \mathcal{C}([a,b])$ such that $Tf^* = f^*$.