Quiz 8

Analysis

April 6, 2018

This is a group assignment. You are encouraged to work together. You can use the board and talk to each other. However, you cannot use the Internet! Jessi is a fantastic resource. Feel free to ask her questions or ask for hints. The point of this assignment is to provide an introduction to discrete dynamical systems.

A discrete dynamical system on $D \subset \mathbb{R}$ is a relationship between iterates $x_n \in D$ given by

$$x_{n+1} = f(x_n),$$

where $f : D \rightarrow D$ is a function. Another commonly used terminology is an iterated map. Some standard definitions in this field are the following:

- The orbit $\gamma(x_0)$ of a point $x_0$ is a set defined by:

$$\gamma(x_0) = \{x_n : x_n = f(x_{n-1})\}.$$

- A period $k$ orbit is an orbit with cardinality $k$:

$$\gamma(x_0) = \{x_0, x_1, \ldots, x_{k-1}\}.$$  

- $x^*$ is a fixed point if $x^* = f(x^*)$.

- A fixed point $x^*$ is locally stable if there exists an open interval $I \subset D$ containing $x^*$ such that for all $x_0 \in I$, $\lim_{n \to \infty} f(x_n) = x^*$.

- A fixed point $x^*$ is globally stable if for all $x_0 \in D$, $\lim_{n \to \infty} f(x_n) = x^*$.

- A fixed point $x^*$ is unstable if it is not locally stable.

1. If $x^*$ is a fixed point of $f$, what is $\gamma(x^*)$?

$$\gamma(x^*) = \{x^*\}$$

such that $x_0$ is a fixed point of $g$.

2. If $\gamma(x_0)$ is a period 2 orbit of $f$, find a function $g$ for which $x_0$ is a fixed point of $g$.

Let $g = f \circ f$. Then,

$$g(x_0) = f(f(x_0)) = f(x_1) = x_0.$$
3. If \( x^* \) is a fixed point of \( f \), prove that if \( |f'(x^*)| < 1 \), then \( x^* \) is locally stable. Hint: This is a contraction mapping type argument.

By the mean value theorem it follows that

\[
|X_n - x^*| = |f(X_n) - f(x^*)| = |f'(c)||X_{n-1} - x^*|
\]

If \( |f'(x^*)| < 1 \), it follows from continuity that there exists \( I \) containing \( x^* \) such that for all \( c \in I \), \( |f'(c)| < 1 \). Therefore on \( I \), \( f \) is a contraction which proves stability.

4. Find and classify the fixed points as stable or unstable for the following iterated map on \( \mathbb{R} \):

\[
x_{n+1} = \frac{3}{2} x_n (1 - x_n).
\]

Fixed points:

\[
X = 0, \quad 1 = \frac{3}{2} (1 - X) \quad \Rightarrow X = \frac{1}{3}
\]

\[
f'(x) = \frac{3}{2} - 3x
\]

\[
f'(0) = \frac{3}{2}, \quad f'(\frac{1}{3}) = \frac{1}{2}
\]

0 is unstable
\( \frac{1}{3} \) is stable.
5. Come up with a definition for what it means for a period $k$ orbit to be locally stable. **Hint:** Thinking of compositions of $f$ might be useful.

A period $k$ orbit $\gamma(x_0) = \{x_0, \ldots, x_{k-1}\}$ is stable if $x_0$ is a stable fixed point of $f^k(x)$.

6. Suppose $\gamma(x_0)$ is a period $k$ orbit. Prove that $\gamma(x_0)$ is locally stable if

$$\prod_{i=0}^{k-1} |f'(x_i)| < 1.$$ 

$$\frac{d}{dx} f^k(x) = f'(x_{k-1}) \cdot f'(x_{k-2}) \cdot \cdots \cdot f'(x_0) < 1.$$
7. Completely analyze the following iterated map on $[0, 1]$: \[
x_{n+1} = \begin{cases} 
2x_n & \text{if } 0 \leq x_n \leq \frac{1}{2} \\
2 - 2x_n & \text{if } \frac{1}{2} \leq x_n \leq 1
\end{cases}
\]
i.e. determine the existence and stability of any fixed points or periodic orbits. Is there an attracting set for this problem? If so, what is it? Hint: Drawing $f$ and its compositions could be very useful for this problem.

All fixed points and periodic orbits are unstable. Attracting set is a topological Cantor set.