Quiz 9

Analysis

April 9, 2018

This is a group assignment. You are encouraged to work together. You can use the board and talk to each other. However, you cannot use the Internet! Jessi is a fantastic resource. Feel free to ask her questions or ask for hints. The point of this assignment is learn how to construct smooth functions with compact support. This will constitute your lecture on this topic.

1. Consider the following function:

$$\varphi(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

- Show that φ is continuous at $x = 0$.

$$\lim_{x \to 0} \varphi(x) = \lim_{x \to 0} e^{-1/x^2} = \lim_{y \to \infty} e^{-y} = 0 = \varphi(0)$$

- Show that φ' exists and $\varphi'(0) = 0$.

$$\lim_{h \to 0} \frac{\varphi(h) - \varphi(0)}{h} = \lim_{h \to 0} \frac{e^{-1/h^2}}{h}$$

$$= \lim_{y \to \infty} y e^{-y^2}$$

$$= \lim_{y \to \infty} \frac{y}{e^{y^2}}$$

$$= \lim_{y \to \infty} \frac{1}{2y e^{y^2}}$$

$$= 0$$

- Prove that for all $k \in \mathbb{N}$ and $x \neq 0$,

$$\varphi^{(k)}(x) = \begin{cases} p_k\left(\frac{1}{x}\right) e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases},$$

  where p_k is a polynomial. This proves that φ is a smooth function.

We do this by induction. Suppose

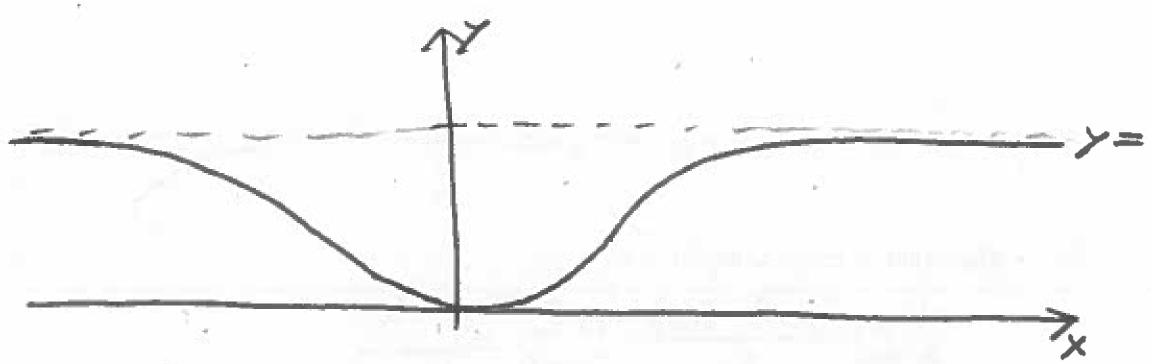
$$\varphi^{(k)}(x) = \begin{cases} p_k\left(\frac{1}{x}\right) e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$\Rightarrow \varphi^{(k+1)}(x) = p_k'\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)e^{-1/x^2} + p_k\left(\frac{1}{x}\right)\left(-\frac{2}{x^3}\right)e^{-\frac{1}{x^2}}$$

$$= \left[p_k'\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) + p_k\left(\frac{1}{x}\right)\left(-\frac{2}{x^3}\right)\right]e^{-1/x^2}$$

$$= p_{k+1}\left(\frac{1}{x}\right)e^{-1/x^2}.$$

- Sketch a graph of $\varphi(x)$.



- Find the Taylor series for $\varphi(x)$ about $x = 0$. How does this result not contradict the fact that φ is smooth?

The Taylor series is identically 0. This is because φ is not analytic. That is, the radius of convergence of the Taylor series about $x=0$ is identically 0.

2. A **smooth bump function** with compact support on $[a, b]$ is a smooth function satisfying:

$$f(x) = \begin{cases} 0, & x \leq a \\ \in (0,1), & a < x < b \, . \\ 0, & x \geq b \end{cases}$$

Show that a smooth bump function can be constructed by appropriately modifying the following function:

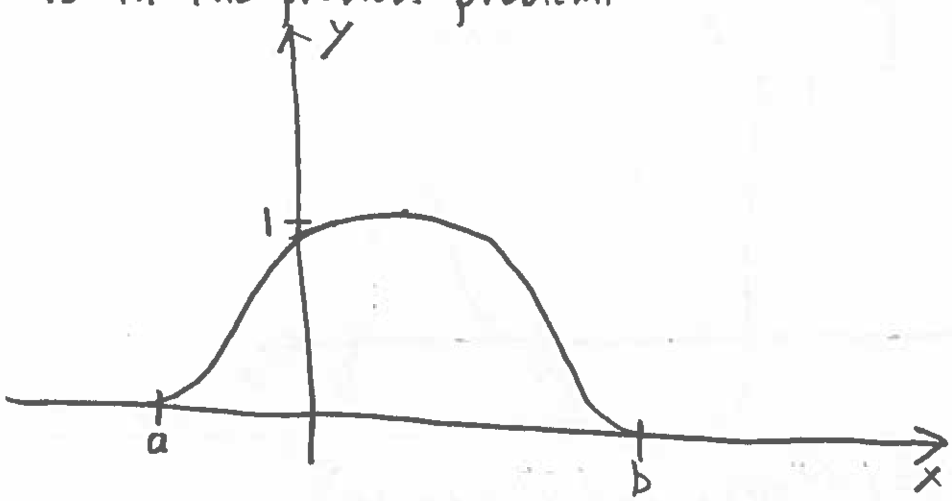
$$\varphi(x) = e^{-1/(x-a)(x-b)}.$$

Sketch a graph of your smooth bump function.

Define

$$\varphi_b(x) = \begin{cases} 0, & x \leq a \\ \varphi(x), & a < x < b \\ 0, & x \geq b \end{cases}$$

The smoothness follows from the same type of arguments as in the previous problem.



3. Sometimes a refinement of the construction of a smooth bump function is required. Let $0 < r < R$. Construct a C^∞ function Ψ satisfying:

$$\Psi(x) = \begin{cases} 1, & \text{if } |x| < r \\ \in [0,1], & \text{if } r < |x| < R \\ 0, & \text{if } R < |x| \end{cases}.$$

Hint: Let

$$\psi_1(x) = \begin{cases} e^{-1/(x-r)} e^{-1/(R-x)}, & \text{if } r < x < R \\ 0, & \text{otherwise} \end{cases}.$$

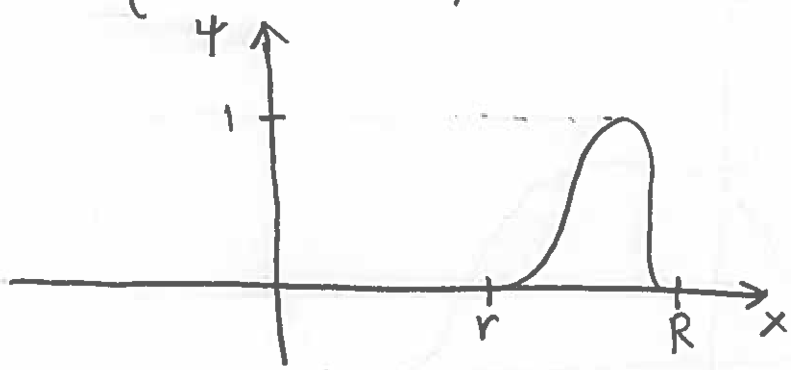
Then let

$$\psi_2(x) = \left(\int_x^R \psi_1(t)\,dt \right) \left(\int_r^R \psi_1(t)\,dt \right)^{-1}.$$

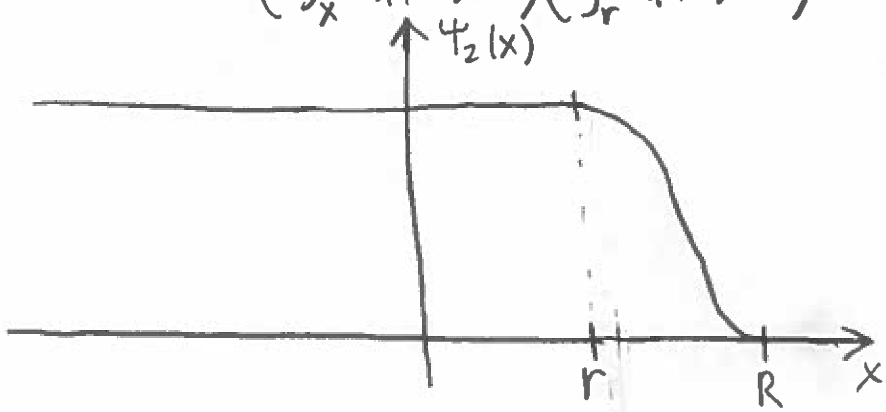
Finally, define $\Psi(x) = \psi_2(|x|)$.

Let

$$\psi_1(x) = \begin{cases} e^{-1/(x-r)(R-x)}, & \text{if } r<x<R \\ 0, & \text{o.w.} \end{cases}$$



Let $\psi_2(x) = \left(\int_x^R \psi_1(t)\,dt \right) \left(\int_r^R \psi_1(t)\,dt \right)^{-1}.$

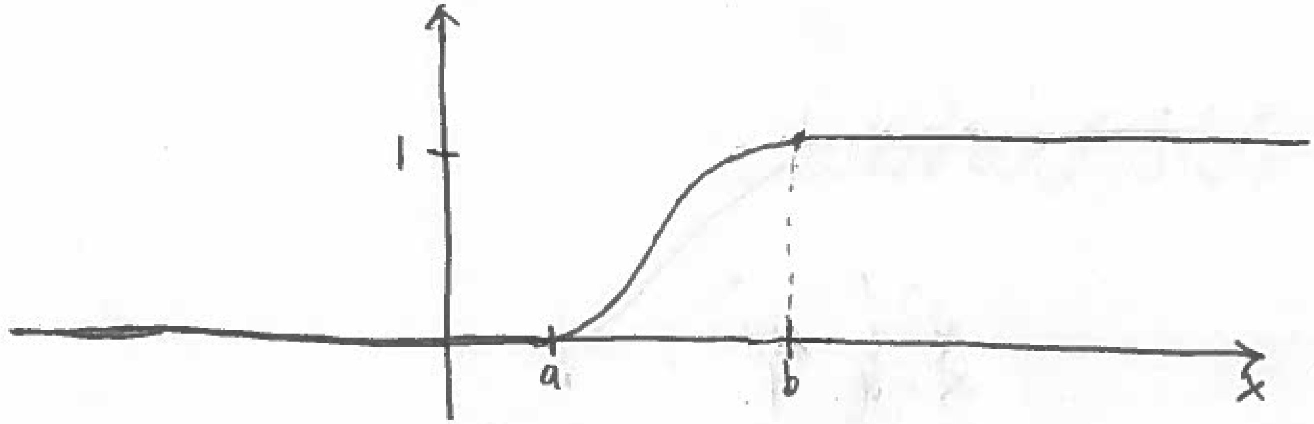


Finally, if we set $\Psi(x) = \psi_2(|x|)$, the result follows.

4. Let φ be a smooth bump function on $[a, b]$ as constructed in problem #2. Define,

$$\Phi(x) = \left(\int_a^b \varphi(t)\, dt \right)^{-1} \int_{-\infty}^{x} \varphi(t)\, dt.$$

Sketch a graph of $\Phi(x)$. Show that Φ is smooth.



5. In many applications, one wants to construct a **smooth transition layer of width α between to states c and d**. This is a function $\chi \in C^\infty$ satisfying:

$$\chi(x) = \begin{cases} c & \text{if } x \le a - \alpha \\ \in [c, d] & \text{if } a - \alpha \le x \le a + \alpha \\ d & \text{if } x \ge a + \alpha \end{cases}.$$

Find and sketch the graph of such a function. **Hint:** Use the function Φ from the proceeding problem.

Let

$\chi(x) = (d-c)\Phi(x) + c$

where Φ is chosen as above with $b - a = 2\alpha$ and $a = a - \alpha$.