MTH 225
Spring 2025
Exam 1
02/17/25

Name (Print): Key

This exam contains 7 pages (including this cover page) and 7 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Short answer questions:** Questions labeled as **"Short Answer"** can be answered by simply writing an equation or a sentence or appropriately drawing a figure. No calculations are necessary or expected for these problems.
- **Unless the question is specified as short answer, mysterious or unsupported answers might not receive full credit.** An incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

Problem	Points	Score
1	10	
2	15	
3	10	
4	15	
5	15	
6	20	
7	15	
Total:	100	

Do not write in the table to the right.

1. (10 points) (**Short Answer**) Determine if the following statement is correct (**C**) or incorrect (**I**). Just circle **C** or **I**. No need to show any work. In order for a statement to be correct it must be true in all cases.

   C **I** If $z_1, z_2 \in \mathbb{C}$, then $\mathrm{Re}(z_1 z_2) = \mathrm{Re}(z_1)\mathrm{Re}(z_2)$.

   C **I** If $z_1, z_2 \in \mathbb{C}$, then $|z_1 z_2| = |z_1| \cdot |z_2|$.

   C **I** If $z = 3\pi i$ then $e^z = -1$.

   C **I** If $A, B \in M_{2\times 2}(\mathbb{R})$ and $AB = I$, where I denotes the 2×2 identity matrix, then $A = B^{-1}$.

   C **I** If $A, B \in M_{2\times 2}(\mathbb{R})$ and $AB = 0$, where 0 denotes the 2×2 zero matrix, then $A = 0$ or $B = 0$.

2. (15 points) For $a \in \mathbb{R}$ consider the following system of equations:

$$\begin{aligned} ax + y + az &= 1, \\ y + z &= 1, \\ (a+1)z &= 1. \end{aligned}$$

   (a) (5 points) Write down the augmented matrix that corresponds to this system of equations.

$$\left[\begin{array}{ccc|c} a & 1 & a & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & a+1 & 1 \end{array}\right]$$

   (b) (10 points) For what values of $a \in \mathbb{R}$ does this system of equations have zero, one, or infinitely many solutions.

- Unique solution if $a \neq 0$ and $a \neq -1$
- No solution if $a = -1$
- No solution if $a = 0$

$\det(A) = a(a+1)$

3. (10 points) Find all solutions to the following system of equations over $\mathbb{Z}_3$:

$$w + x + y + 2z = 1,$$
$$2x + 2y + z = 0,$$
$$2w + 2y + z = 1.$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 2 & 1 \\ 0 & 2 & 2 & 1 & 0 \\ 2 & 0 & 2 & 1 & 1 \end{array}\right]\begin{matrix} \\ \\ +R1 \end{matrix} \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 2 & 1 \\ 0 & 2 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{array}\right]\begin{matrix} \\ \updownarrow \\ \end{matrix} \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 2 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 2 & 2 & 1 & 0 \end{array}\right]\begin{matrix} \\ \\ +R2 \end{matrix}$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 2 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 2 & 1 & 2 \end{array}\right]\begin{matrix} +R3 \\ \\ \\ \end{matrix} \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 2 & 1 & 2 \end{array}\right]\begin{matrix} +2R2 \\ \\ \times 2 \end{matrix} \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 2 & 1 \end{array}\right]$$

Therefore,

$w=1$
$x=2$
$y+2z=1$

$\Rightarrow w=1$
$x=2$
$y=1+z$

The solutions are therefore,

$w=1$	$w=1$	$w=1$
$x=2$	$x=2$	$x=2$
$y=1$	$y=2$	$y=0$
$z=0$	$z=1$	$z=2$

4. (15 points) (a) (5 points) **Short Answer:** If V is a vector space over a field F and $W \subset V$, write down what you must show in order to prove that W is a subspace of V.

For all $\vec{u}, \vec{v} \in W$ and $\lambda \in F$ we need to show $\vec{u} + \lambda \vec{v} \in W$.

(b) (10 points) If $V = C^2(\mathbb{R})$, and

$$W = \{f \in V : f''(x) - f(x) = 0\},$$

prove that W is a vector space by proving that it is a subspace of V.

Let $f, g \in V$ and for $\lambda \in \mathbb{R}$ let $h(x) = f(x) + \lambda g(x)$.

Consequently,

$$\begin{aligned} h''(x) - h(x) &= f''(x) + \lambda g''(x) - f(x) - \lambda g(x) \\ &= f''(x) - f(x) + \lambda (g''(x) - g(x)) \\ &= 0 \end{aligned}$$

Therefore, $h \in W$.

5. (15 points) Let V be a vector space over a field F.

(a) (5 points) **Short Answer:** Write down the definition of what it means for the three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ to be linearly independent in V.

The only solution to the equation

$$c_1\vec{u} + c_2\vec{v} + c_3\vec{w} = \vec{0}$$

is $c_1 = c_2 = c_3 = 0$.

(b) (5 points) If $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \in V$ are linearly independent, show that the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in V$ given by

$$\mathbf{v}_1 = \mathbf{u}_1 - \mathbf{u}_2, \ \mathbf{v}_2 = \mathbf{u}_2 - \mathbf{u}_3, \ \mathbf{v}_3 = \mathbf{u}_3 + \mathbf{u}_1$$

are linearly independent.

Suppose $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = 0$

$$\Rightarrow (c_1 + c_3)\vec{u}_1 + (-c_1 + c_2)\vec{u}_2 + (-c_2 + c_3)\vec{u}_3 = 0$$

Therefore,

$c_3 = c_2$

$c_2 = c_1$

$c_1 = -c_3$

which is only satisfied if $c_1 = c_2 = c_3 = 0$.

(c) (5 points) If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ are linearly independent, show that the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in V$ given by

$$\mathbf{v}_1 = \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3, \ \mathbf{v}_2 = \mathbf{u}_2 + \mathbf{u}_3, \ \mathbf{v}_3 = \mathbf{u}_1$$

are linearly dependent.

$\vec{v}_1 = \vec{v}_2 + \vec{v}_3$.

Therefore, the vectors are linearly dependent.

6. (20 points) (a) (5 points) **Short Answer:** Let V and W be vector spaces over a field F. Write down what you must show in order to prove that a mapping $T : V \mapsto W$ is a linear transformation.

We need to show for all $\vec{u}, \vec{v} \in V$ and $\lambda \in F$ that

$$T(\vec{u} + \lambda \vec{v}) = T(\vec{u}) + \lambda T(\vec{v}).$$

(b) (10 points) With respect to the standard basis, find the matrix representation of a linear transformation $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$ that satisfies

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix} \text{ and } T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) = \begin{bmatrix}1\\2\end{bmatrix}$$

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \frac{1}{2}T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) + \frac{1}{2}T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) = \begin{bmatrix}1\\1/2\end{bmatrix} + \begin{bmatrix}1/2\\1\end{bmatrix} = \begin{bmatrix}3/2\\3/2\end{bmatrix},$$

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \frac{1}{2}T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) - \frac{1}{2}T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) = \begin{bmatrix}1\\1/2\end{bmatrix} - \begin{bmatrix}1/2\\1\end{bmatrix} = \begin{bmatrix}1/2\\-1/2\end{bmatrix}.$$

Therefore,

$$[T] = \begin{bmatrix}3/2 & 1/2\\3/2 & -1/2\end{bmatrix}$$

(c) (5 points) **Short Answer:** Suppose $T : \mathbb{R}^3 \mapsto \mathbb{R}^3$ is a linear transformation satisfying

$$T\left(\begin{bmatrix}2\\\pi\\\sqrt{2}\end{bmatrix}\right) = \begin{bmatrix}1\\0\\0\end{bmatrix} \text{ and } T\left(\begin{bmatrix}\pi^2\\1\\\sqrt{3}\end{bmatrix}\right) = \begin{bmatrix}0\\1\\0\end{bmatrix} \text{ and } T\left(\begin{bmatrix}8\\2\\17\end{bmatrix}\right) = \begin{bmatrix}0\\0\\1\end{bmatrix}.$$

With respect to the standard basis, find a matrix representation of T^{-1}.

$$[T^{-1}] = \begin{bmatrix}2 & \pi^2 & 8\\\pi & 1 & 2\\\sqrt{2} & \sqrt{3} & 17\end{bmatrix}$$

7. (15 points) Let $T : P_2(\mathbb{R}) \mapsto M_{2\times 2}(\mathbb{R})$ be defined by

$$T(a_0 + a_1 x + a_2 x^2) = \begin{bmatrix} a_1 & 0 \\ a_1 & 2a_2 \end{bmatrix}$$

and let

$$\mathcal{A} = \{1, x, x^2\} \text{ and } \mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

be the standard bases for $P_2(\mathbb{R})$ and $M_{2\times 2}(\mathbb{R})$ respectively.

(a) (5 points) Show that T is a linear transformation.

Let $p_1(x) = a_0 + a_1 x + a_2 x^2$ and $p_2(x) = b_0 + b_1 x + b_3 x^2$ and $\lambda \in \mathbb{R}$.

Therefore,

$$T[p_1 + \lambda p_2] = \begin{bmatrix} a_1 + \lambda b_1 & 0 \\ a_1 + \lambda b_1 & 2(a_2 + \lambda b_2) \end{bmatrix} = \begin{bmatrix} a_1 & 0 \\ a_1 & 2a_2 \end{bmatrix} + \lambda \begin{bmatrix} b_1 & 0 \\ b_1 & 2b_2 \end{bmatrix} = T[p_1] + \lambda T[p_2].$$

(b) (10 points) Find the matrix representation of T with respect to $\mathcal{A}$ and $\mathcal{B}$, i.e., find $[T[\mathcal{A}, \mathcal{B}]]$.

$$T[1] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$T[x] = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = 1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$T[x^2] = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} = 2\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore,

$$[T[\mathcal{A}, \mathcal{B}]] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$