MTH 225
Homework #1

Due Date: January 22, 2025

1. How should the coefficients $a, b, c \in \mathbb{R}$ be chosen so that the systems $ax + by + cz = 3$, $ax - y + cz = 1$, and $x + by - cz = 2$, has the solution $x = 1$, $y = 2$, and $z = -1$.

2. For what values of x, y, z, w are the matrices

$$A = \begin{bmatrix} x+y & x-z \\ y+w & x+2w \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

equal?

3. Consider the following system of equations for $x, y, z \in \mathbb{R}$:

$$\begin{aligned} 2 &= ax + bz, \\ b &= ax + 2y + az, \\ a &= bx + 2y + az. \end{aligned}$$

   (a) For what values of $a, b \in \mathbb{R}$ does this system have a unique solution?
   (b) For what values of $a, b \in \mathbb{R}$ does this system have infinitely many solutions?
   (c) For what values of $a, b \in \mathbb{R}$ does this system have no solution?

4. Consider the following matrices $A, B, C \in M_{2\times 2}(\mathbb{R})$:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}.$$

   (a) Show that $AB = AC$ even though $B \neq C$.
   (b) Show that $B^2 - C^2 \neq (B - C)(B + C)$.

5. A matrix $S \in M_{n\times n}(\mathbb{R})$ is said to be a square root of the matrix $A \in m_{n\times n}(\mathbb{R})$ if $S^2 = A$.

   (a) Show that $S = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$ is a square root of the matrix $A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$.
   (b) Find all real square roots of the 2×2 identity matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or prove that none exist.
   (c) Find all real square roots of the matrix $-I = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ or prove that none exist.

6. A matrix $A \in M_{n\times n}(\mathbb{R})$ is called idempotent if $A^2 = A$.

   (a) Show that $A \in M_{n\times n}(\mathbb{R})$ is idempotent if and only if $I - A$ is idempotent, where $I \in M_{n\times n}(\mathbb{R})$ is the $n \times n$ identity matrix.

   (b) Suppose $A, B \in M_{n\times n}(\mathbb{R})$ are idempotent. Show that $(A - B)^3 = A - B$ or explain why it is not true.

   (c) Suppose $A \in M_{n\times n}(\mathbb{R})$ is idempotent. Show that A is invertible if and only if $A = I$.

7. Suppose $A \in M_{n\times n}(\mathbb{R})$ satisfies $A^2 - 3A + I = 0$, where $0 \in M_{n\times n}(\mathbb{R})$ is the $n \times n$ zero matrix and $I \in M_{n\times n}(\mathbb{R})$ is the identity matrix. Show that $A^{-1} = 3I - A$.

8. For $x, y, z, w \in \mathbb{R}$ and $x_1, \ldots, x_n \in \mathbb{R}$, the Vandermonde matrices are defined by

$$V_1 = \begin{bmatrix}1\end{bmatrix},\ V_2 = \begin{bmatrix}1 & x\\ 1 & y\end{bmatrix},\ V_3 = \begin{bmatrix}1 & x & x^2\\ 1 & y & y^2\\ 1 & z & z^2\end{bmatrix},\ V_4 = \begin{bmatrix}1 & x & x^2 & x^3\\ 1 & y & y^2 & y^3\\ 1 & z & z^2 & z^3\\ 1 & w & w^2 & w^3\end{bmatrix},\ V_n = \begin{bmatrix}1 & x_1 & x_1^2 & \cdots & x_1^{n-1}\\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1}\\ \vdots & \vdots & \vdots & & \vdots\\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1}\end{bmatrix}.$$

   (a) Compute the determinants of V_1, V_2, V_3, and V_4 and simplify your answers as much as possible. **Hint:** Factor as much as possible.

   (b) Find a general formula for the determinant of V_n. You do not have to formally prove this.

   (c) Find conditions on $x_1, x_2, \ldots, x_n$ that are necessary and sufficient for V_n to be invertible.

Homework #1

#1

How should the coefficients $a, b, c \in \mathbb{R}$ be chosen so that the system $ax+by+cz=3$, $ax-y+cz=1$, and $x+by-cz=2$, has the solution $x=1$, $y=2$, and $z=-1$.

Solution:

Setting $x=1, y=2, z=-1$, it follows that

$$\begin{aligned} a+2b-c&=3 & \qquad a+2b-c&=3 \\ a-2-c&=1 \;\Rightarrow & a-c&=3 \\ 1+2b+c&=2 & 2b+c&=1 \end{aligned}$$

This yields the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 1 & 0 & -1 & 3 \\ 0 & 2 & 1 & 1 \end{array}\right] \begin{array}{c} \\ -R1 \\ \\ \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -2 & 0 & 0 \\ 0 & 2 & 1 & 1 \end{array}\right] \begin{array}{c} \\ \\ +R2 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & -2 & -1 & 3 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Therefore,

$c=1$, $b=0$, and $a=4$

$\blacksquare$

#2

For what values of $x, y, z, w \in \mathbb{R}$ are the matrices

$$A = \begin{bmatrix} x+y & x-z \\ y+w & x+2w \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

equal?

Solution:

Setting components equal, we have that

$$\begin{aligned} x+y&=1 \\ x-z&=0 \\ y+w&=2 \\ x+2w&=1 \end{aligned}$$

The corresponding augmented matrix is given by

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 2 \\ 1 & 0 & 0 & 2 & 1 \end{array}\right]\begin{array}{l} \\ -R1 \\ \\ -R1 \end{array} \Rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & -1 & 0 & 2 & 0 \end{array}\right]\begin{array}{l} \\ +R3 \\ \\ +R3 \end{array} \Rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 & 2 \end{array}\right]\begin{array}{l} \\ \updownarrow \times -1 \\ \\ /3 \end{array}$$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 2/3 \end{array}\right]\begin{array}{l} \\ -R4 \\ +R4 \\ \\ \end{array} \Rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 4/3 \\ 0 & 0 & 1 & 0 & -1/3 \\ 0 & 0 & 0 & 1 & 2/3 \end{array}\right]\begin{array}{l} -R2 \\ \\ \\ \\ \end{array} \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1/3 \\ 0 & 1 & 0 & 0 & 4/3 \\ 0 & 0 & 1 & 0 & -1/3 \\ 0 & 0 & 0 & 1 & 2/3 \end{array}\right]$$

$$\Rightarrow x = -1/3,\ y = 4/3,\ z = -1/3,\ w = 2/3.$$

■

#3

Consider the following system of equations for $x, y, z \in \mathbb{R}$:

$$2 = ax + bz$$
$$b = ax + 2y + az$$
$$a = bx + 2y + az$$

(a) For what values of $a, b \in \mathbb{R}$ does this system have a unique solution?

(b) For what values of $a, b \in \mathbb{R}$ does this system have infinitely many solutions?

(c) For what values of $a, b \in \mathbb{R}$ does this system have no solution?

Solution:

We can think of this system as matrix multiplication

$$A\vec{x} = \vec{b},$$

where

$$A = \begin{bmatrix} a & 0 & b \\ a & 2 & a \\ b & 2 & a \end{bmatrix},\quad \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix},\quad \vec{b} = \begin{bmatrix} 2 \\ b \\ a \end{bmatrix}$$

Now, A is invertible if $\det(A) \neq 0$. Calculating, we have that

$$\det(A) = 2b(a-b)$$

Therefore, if $a \neq b$ and $b \neq 0$ then A is invertible and thus the system has the unique solution

$$\vec{x} = A^{-1}\vec{b}.$$

Case 1: If $a=b$ then we have the system

$$\left[\begin{array}{ccc|c} a & 0 & a & 2 \\ a & 2a & a & a \\ a & 2a & a & a \end{array}\right] \begin{array}{l} \\ -R1 \\ -R1 \end{array} \Rightarrow \left[\begin{array}{ccc|c} a & 0 & a & 2 \\ 0 & 2 & 0 & a-2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

If $a \neq 0$ we can row reduce into the form:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & a-2/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore, if $a=b$ and $a \neq 0$ there are infinitely many solutions. If $a=b$ and $a=0$ then we have

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 2 \\ 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{array}\right]$$

which yields the contradiction $2=0$. Therefore, if $a=b=0$ there are no solutions.

Case 2: If $b=0$ then we have the system

$$\left[\begin{array}{ccc|c} a & 0 & 0 & 2 \\ a & 2 & a & 0 \\ 0 & 2 & a & a \end{array}\right] \begin{array}{l} \\ -R1 \\ \end{array} \Rightarrow \left[\begin{array}{ccc|c} a & 0 & 0 & 2 \\ 0 & 2 & a & -2 \\ 0 & 2 & a & a \end{array}\right] \begin{array}{l} \\ \\ -R2 \end{array} \Rightarrow \left[\begin{array}{ccc|c} a & 0 & 0 & 2 \\ 0 & 2 & a & -2 \\ 0 & 0 & 0 & a+2 \end{array}\right]$$

Therefore, if $a=-2$ there are no solutions while if $a \neq -2$ there are infinitely many solutions.

Summary:

(a) If $a \neq b$ and $b \neq 0$, the solution is unique.

(b) If $a=b$ and $a \neq 0$ or if $b=0$ and $a \neq -2$ there are infinitely many solutions.

(c) If $a=b=0$ or $b=0$ and $a=-2$ then there are no solutions. ■

#4

Consider the following matrices $A, B, C \in M_{2\times 2}(\mathbb{R})$:

$$A=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B=\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}, \quad C=\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$

(a) Show that $AB=AC$ even though $B \neq C$.

(b) Show that $B^2-C^2 \neq (B-C)(B+C)$.

Solution:

(a) $A\cdot B=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}=\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$, $A\cdot C=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}=\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$.

(b) $B^2=\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}=\begin{bmatrix} 19 & 18 \\ 6 & 7 \end{bmatrix}$, $C^2=\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}=\begin{bmatrix} 13 & 20 \\ 5 & 8 \end{bmatrix}$

$\Rightarrow B^2-C^2=\begin{bmatrix} 6 & -2 \\ 1 & -1 \end{bmatrix}$

However,

$(B-C)(B+C)=\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 7 & 7 \\ 2 & 4 \end{bmatrix}=\begin{bmatrix} 5 & 3 \\ 0 & 0 \end{bmatrix}$ ■

#5

A matrix $S \in M_{n\times n}(\mathbb{R})$ is said to be a square root of the matrix $A \in M_{n\times n}(\mathbb{R})$ if $S^2 = A$.

(a) Show that $S = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$ is a square root of $A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$.

(b) Find all square roots of the 2x2 identity $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or prove that none exist.

(c) Find all real square roots of the matrix $-I = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ or prove that none exist.

Solution:

(a) $S^2 = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

(b) Let $S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Therefore,

$$S^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+bc & ab+bd \\ ca+dc & cb+d^2 \end{bmatrix}$$

If $S^2 = I$ we must have:

$$a^2+bc=1$$
$$b(a+d)=0$$
$$c(a+d)=0$$
$$cb+d^2=1$$

Case 1:

If $a=-d$ then $a^2+bc=1$. So, as long as we set $a=-d$ then any combination of a, b, c satisfying $a^2+bc=1$ works.

Case 2:

If $b=c=0$ then $a^2=1$ and $d^2=1$ which yields the matrices

$$S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

(c) Again letting $S=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we obtain the following equations

$$a^2+bc=-1$$
$$b(a+d)=0$$
$$c(a+d)=0$$
$$cb+d^2=-1$$

Now the only case is $a=-d$ and we obtain any combination of a,b,c satisfying $a^2+bc=-1$ works.

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#6.

A matrix $A \in M_{n\times n}(\mathbb{R})$ is called idempotent if $A^2=A$.

(a) Show that $A \in M_{n\times n}(\mathbb{R})$ is idempotent if and only if $I-A$ is idempotent.

(c) Suppose $A \in M_{n\times n}(\mathbb{R})$ is idempotent. Show that A is invertible if and only if $A=I$.

Solution:

(a) ($\Rightarrow$) If A is idempotent then

$$(I-A)^2=I-2A+A^2=I-2A+A=I-A.$$

($\Leftarrow$) If $I-A$ is idempotent then

$$I-2A+A^2=I-A$$
$$\Rightarrow A^2=A.$$

(c) ($\Rightarrow$) If A is invertible, then since $A^2=A$ it follows that

$$A^2=A$$
$$\Rightarrow A^{-1}\cdot A\cdot A=A^{-1}A$$
$$\Rightarrow A=I.$$

($\Leftarrow$) Now, if $A=I$ then A is invertible with inverse $A^{-1}=I$.

$\blacksquare$

#7

Suppose $A \in M_{n\times n}(\mathbb{R})$ satisfies $A^2-3A+I=0$. Show that

$$A^{-1}=3I-A.$$

Solution:

Since $A^2-3A+I=0$ it follows that $I=3A-A^2$. Therefore

$$A(3I-A)=3A-A^2=I.$$

Consequently, $3I-A=A^{-1}$.

#8

For $x,y,z \in \mathbb{R}$ and $x_1,\dots,x_n \in \mathbb{R}$ the Vandermonde matrices are defined by

$$V_1=[1],\ V_2=\begin{bmatrix}1 & x\\ 1 & y\end{bmatrix},\ V_3=\begin{bmatrix}1 & x & x^2\\ 1 & y & y^2\\ 1 & z & z^2\end{bmatrix},\ V_4=\begin{bmatrix}1 & x_1 & \cdots & x_1^{n-1}\\ 1 & x_2 & \cdots & x_2^{n-1}\\ \vdots & & & \vdots\\ 1 & x_n & \cdots & x_n^{n-1}\end{bmatrix}$$

(a) Compute determinants of V_1, V_2, V_3, V_4.

(b) Find a general formula for the determinant of V_n.

(c) Find conditions on $x_1,\dots,x_n$ that are necessary and sufficient for V_n to be invertible.

Solution:

(a) • $\det(V_1)=1$

• $\det(V_2)=\det\left(\begin{bmatrix}1 & x\\ 1 & y\end{bmatrix}\right)=\det\left(\begin{bmatrix}1 & x\\ 0 & y-x\end{bmatrix}\right)=y-x.$

• $\det(V_3)=\det\left(\begin{bmatrix}1 & x & x^2\\ 1 & y & y^2\\ 1 & z & z^2\end{bmatrix}\right)$

$$=\det\left(\begin{bmatrix}1 & x & x^2\\ 0 & y-x & y^2-x^2\\ 0 & z-x & z^2-x^2\end{bmatrix}\right)$$

$$=\det\left(\begin{bmatrix}1 & x & x^2\\ 0 & y-x & (y-x)(y+x)\\ 0 & z-x & (z-x)(z+x)\end{bmatrix}\right)$$

$$=(z-x)(y-x)\det\left(\begin{bmatrix}1 & x & x^2\\ 0 & 1 & y+x\\ 0 & 1 & z+x\end{bmatrix}\right)$$

$$\Rightarrow \det(V_3) = (z-x)(y-x)\det\left(\begin{bmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 0 & z-y \end{bmatrix}\right)$$

$$= (z-x)(z-y)(y-x)\det\left(\begin{bmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$= (z-x)(z-y)(y-x)$$

- $$\det(V_4) = \det\left(\begin{bmatrix} 1 & x & x^2 & x^3 \\ 1 & y & y^2 & y^3 \\ 1 & z & z^2 & z^3 \\ 1 & w & w^2 & w^3 \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} 1 & x & x^2 & x^3 \\ 0 & y-x & y^2-x^2 & y^3-x^3 \\ 0 & z-x & z^2-x^2 & z^3-x^3 \\ 0 & w-x & w^2-x^2 & w^3-x^3 \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} 1 & x & x^2 & x^3 \\ 0 & y-x & (y-x)(y+x) & (y-x)(y^2+yx+x^2) \\ 0 & z-x & (z-x)(z+x) & (z-x)(z^2+zx+x^2) \\ 0 & w-x & (w-x)(w+x) & (w-x)(w^2+wx+x^2) \end{bmatrix}\right)$$

$$= (w-x)(z-x)(y-x)\det\left(\begin{bmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & y+x & y^2+yx+x^2 \\ 0 & 1 & z+x & z^2+zx+x^2 \\ 0 & 1 & w+x & w^2+wx+x^2 \end{bmatrix}\right)$$

$$= (w-x)(z-x)(y-x)\det\left(\begin{bmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & y+x & y^2+yx+x^2 \\ 0 & 0 & z-y & z^2+zx-y^2-yx \\ 0 & 0 & w-y & w^2+wx-y^2-yx \end{bmatrix}\right)$$

$$= (w-x)(z-x)(y-x)\det\left(\begin{bmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & y+x & y^2+yx+x^2 \\ 0 & 0 & z-y & (z-y)(z+x) \\ 0 & 0 & w-y & (w-y)(w+x) \end{bmatrix}\right)$$

$$= (w-x)(z-x)(y-x)(z-y)(w-y)\det\left(\begin{bmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & y+x & y^2+yx+x^2 \\ 0 & 0 & 1 & z+x \\ 0 & 0 & 1 & w+x \end{bmatrix}\right)$$

$$= (w-x)(z-x)(y-x)(z-y)(w-y)\det\left(\begin{bmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & y+x & y^2+yx+x^2 \\ 0 & 0 & 1 & z+x \\ 0 & 0 & 0 & w-z \end{bmatrix}\right)$$

$$= (w-x)(z-x)(y-x)(z-y)(w-y)(w-z).$$

(b) Assuming this pattern continues,

$$\det(V_n) = (x_n - x_{n-1})\cdot(x_n - x_{n-2})\cdots(x_n - x_1)\cdot$$
$$\times$$
$$(x_{n-1} - x_{n-2})(x_{n-1} - x_{n-3})\cdots(x_{n-1} - x_1)$$
$$\times$$
$$\vdots$$
$$\times$$
$$(x_2 - x_1)$$

(c) For all $i,j \in \{1,\dots,n\}$, $x_i \neq x_j$ if $i \neq j$.