MTH 225
Homework #2

Due Date: January 29, 2025

1. If $z = 1 + 2i$, $w = 2 - i$, and $\zeta = 4 + 3i$, write the following complex expressions in the form $a + bi$, where $a, b \in \mathbb{R}$.
   (a) $z + 3w$
   (b) $-2w + \overline{\zeta}$
   (c) z^2
   (d) $w^3 + w$
   (e) $\mathrm{Im}(\zeta^{-1})$
   (f) w/z
   (g) $\zeta^2 + 2\overline{\zeta} + 3$

2. Write the given complex number in the form $a + bi$, where $a, b \in \mathbb{R}$.
   (a) i^{101}
   (b) $\frac{1+2i}{5-3i}$
   (c) $e^{i\pi/6}$
   (d) $(1 + i\sqrt{3})^{10}$

2 3. Write the given complex number in the form $a + bi$, where $a, b \in \mathbb{R}$.
   1/2 (a) $e^{-i\pi/2}$
   1/2 (b) $\dfrac{e^{1+3\pi i}}{e^{-1+\pi/2i}}$
   1/2 (c) $\frac{e^{3i} - e^{-3i}}{2i}$
   1/2 (d) e^{e^i}

4. Let $z \in \mathbb{C}$ and assume $z \neq 0$. Prove the following
   (a) $|\mathrm{Re}(z)| \leq |z|$ and $|\mathrm{Im}(z)| \leq |z|$.
   (b) $\mathrm{Re}(z) = (z + \overline{z})/2$ and $\mathrm{Im}(z) = -i(z - \overline{z})/2$.
   (c) $|z| = 1$ if and only if $1/z = \overline{z}$.
   (d) If $|z| = 1$ and $z \neq 1$, then $\mathrm{Re}((1 - z)^{-1}) = 1/2$.

don't do.

5. Compute the following

   (a) $5 + 3^4$ in $\mathbb{Z}_{11}$.
   (b) 7^{-1} in $\mathbb{Z}_{13}$.
   (c) 3^{101} in $\mathbb{Z}_5$.
   (d) -16 in $\mathbb{Z}_{19}$.
   (e) $7 \cdot (3 + 13)$ in $\mathbb{Z}_{11}$.
   (f) 8^{-1} in $\mathbb{Z}_{15}$.

2 6. Find the inverse of the following matrix:

$$\begin{bmatrix} 2-3i & 4 \\ 1 & 2 \end{bmatrix}.$$

7. Show that 10 does not have a multiplicative inverse in $\mathbb{Z}_{15}$.

2 8. Find all solutions to the equation $x^2 = 1$ in $\mathbb{Z}_8$.

9. Find a quadratic polynomial $x^2 + bx + c$ over $\mathbb{Z}_6$ which has four distinct roots in $\mathbb{Z}_6$.

10. Find two nonzero elements in $\mathbb{Z}_{14}$ whose product is 0.

11. Solve the following system of linear equations over $\mathbb{Z}_3$:

$$\begin{aligned} 2x + y &= 1, \\ 2x + 2y &= 2. \end{aligned}$$

2 12. Find all solutions to the following system of linear equations over $\mathbb{Z}_5$:

$$\begin{aligned} 2w + 2x + y + z &= 4, \\ w + x + 3y + z &= 1, \\ x + y &= 1. \end{aligned}$$

Remember, in $\mathbb{Z}_5$ you should only have a finite number of solutions.

13. Find the inverse of the following matrix over $\mathbb{Z}_3$:

2

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

Homework #2

#1

If $z = 1+2i$, $w = 2-i$, and $\xi = 4+3i$, write the following expressions in the form $a+bi$, where $a,b \in \mathbb{R}$.

(a) $z+3w$

(b) $-2w+\bar{\xi}$

(c) z^2

(d) w^3+w

(e) $\mathrm{Im}(\xi^{-1})$

(f) w/z

(g) $\xi^2+2\bar{\xi}+3$.

Solution:

(a) $z+3w = 1+2i+3(2-i) = 1+2i+6-3i = 7-i$.

(b) $-2w+\bar{\xi} = -2(2-i)+4-3i = -4+2i+4-3i = -i$.

(c) $z^2 = (1+2i)^2 = 1+4i-4 = -3+4i$

(d) $w^3+w = w(w^2+1) = w(w+i)(w-i) = (2-i)\cdot 2\cdot(2-2i)$

$\Rightarrow w^3+w = 4(2-i)(1-i) = 4\cdot(2-3i-1) = 4(1-3i)$

(e) $\mathrm{Im}(\xi^{-1}) = \mathrm{Im}((4+3i)^{-1}) = \mathrm{Im}\left(\dfrac{1}{4+3i}\right) = \mathrm{Im}\left(\dfrac{4-3i}{25}\right) = -\dfrac{3}{25}$.

(f) $\dfrac{w}{z} = \dfrac{2-i}{1+2i} = \dfrac{(2-i)(1-2i)}{5} = -\dfrac{3i}{5}$

(g) $\xi^2+2\bar{\xi}+3 = (4+3i)^2+2(4-3i)+3$

$= 16+24i-9+8-6i+3$

$= 18-18i$

$\blacksquare$

#2

Write the given complex number in the form $a+bi$, where $a,b \in \mathbb{R}$.

(a) i^{101}

(b) $\frac{1+2i}{5-3i}$

(c) $e^{i\pi/6}$

(d) $(1+i\sqrt{3})^{10}$

Solution:

(a) $i^{101} = (i^4)^{25} \cdot i = i$

(b) $\frac{1+2i}{5-3i} = \frac{(1+2i)(5-3i)}{34} = \frac{11}{34} + \frac{7}{34}i$

(c) $e^{i\pi/6} = \cos(\pi/6) + i\sin(\pi/6)$
$= \sqrt{3}/2 + 1/2\, i$

(d) $(1+i\sqrt{3})^{10} = 2^{10}(1/2 + i\sqrt{3}/2)^{10}$
$= 2^{10}(e^{i\pi/3})^{10}$
$= 2^{10}(\cos(10\pi/3) + i\sin(10\pi/3))$
$= 2^{10}(\cos(4\pi/3) + i\sin(4\pi/3))$
$= -2^{10}(\cos(\pi/3) + i\sin(\pi/3))$
$= -2^{10}(1/2 + i\sqrt{3}/2)$

#3

Write the given complex number in the form $a+bi$, where $a,b \in \mathbb{R}$.

(a) $e^{-i\pi/2}$

(b) $\frac{e^{1+3\pi i}}{e^{-1+\pi/2\, i}}$

(c) $\frac{e^{3i}-e^{-3i}}{2i}$

(d) $e^{e^{i}}$

Solution:

(a) $e^{-i\pi/2} = \cos(-\pi/2) + i\sin(-\pi/2) = -i$.

(b) $\dfrac{e^{1+3\pi i}}{e^{-1+\pi/2 i}} = e^2 \dfrac{e^{\pi i}}{e^{\pi/2 i}} = e^2 e^{i\pi/2} = ie^2$.

(c) $\dfrac{e^{3i} - e^{-3i}}{2i} = \sin(3)$.

(d) $e^{e^i} = e^{\cos(1) + i\sin(1)} = e^{\cos(1)} e^{i\sin(1)} = e^{\cos(1)}(\cos(\sin(1)) + i\sin(\sin(1)))$. ■

#4

Let $z \in \mathbb{C}$ and assume $z \neq 0$. Prove the following

(a) $|Re(z)| \leq |z|$. and $|Im(z)| \leq |z|$

(b) $Re(z) = \dfrac{z + \bar{z}}{2}$ and $Im(z) = \dfrac{-i(z - \bar{z})}{2}$

(c) $|z| = 1$ if and only if $1/z = \bar{z}$.

(d) If $|z| = 1$ and $z \neq 1$, then $Re((1-z)^{-1}) = 1/2$.

Solution:

(a) Let $z = x + iy$, where $x, y \in \mathbb{R}$. Therefore,

$$|z| = \sqrt{x^2 + y^2} \geq \sqrt{x^2} = |Re(z)|,$$
$$|z| = \sqrt{x^2 + y^2} \geq \sqrt{y^2} = |Im(z)|.$$

(b) Let $z = x + iy$, where $x, y \in \mathbb{R}$. Therefore,

$$\frac{z + \bar{z}}{2} = \frac{x + iy + x - iy}{2} = x = Re(z).$$

$$\frac{-i(z - \bar{z})}{2} = \frac{-i(x + iy - x + iy)}{2} = \frac{-i \cdot 2iy}{2} = y =: Im(z).$$

(c) ($\Rightarrow$) If $|z| = 1$, then there exists $\theta \in \mathbb{R}$ such that $z = e^{i\theta}$. Consequently, $\bar{z} = e^{-i\theta} = 1/z$.

($\Leftarrow$) If $\bar{z} = 1/z$, since $z = |z|e^{i\theta}$ for some $\theta \in \mathbb{R}$ it follows that

$$|z|e^{-i\theta} = 1/|z|e^{-i\theta}$$

and thus

$$|z|^2=1$$

which implies $|z|=1$.

(d) Suppose $|z|=1$ and $z\neq 1$. Therefore, there exists $\theta\in(0,2\pi)$ such that $z=e^{i\theta}$. Consequently,

$$\frac{1}{1-z}=\frac{1}{1-e^{i\theta}}=\frac{1-e^{-i\theta}}{2}=\frac{1-\cos\theta}{2}+\frac{i\sin\theta}{2}.$$

#5

Compute the following

(a) $5+3^4$ in $\mathbb{Z}_{11}$

(b) 7^{-1} in $\mathbb{Z}_{13}$

(c) 3^{101} in $\mathbb{Z}_5$

(d) -16 in $\mathbb{Z}_{19}$

(e) $7\cdot(3+13)$ in $\mathbb{Z}_{11}$

(f) 8^{-1} in $\mathbb{Z}_{15}$

Solution:

(a) $5+3^4=5+81=-2=9$

(b) Since $7\cdot 2=14=1$, it follows that $7^{-1}=2$.

(c) $3^{101}=(3^{100})\cdot 3=0\cdot 3=0$

(d) Since $16+3=19=0$, it follows that $-16=3$.

(e) $7\cdot(3+13)=7\cdot(3+2)=35=2$

(f) Since $8\cdot 2=16=1$, it follows that $8^{-1}=2$.

#6.

Find the inverse of the following matrix

$$A=\begin{bmatrix} 2-3i & 4 \\ 1 & 2 \end{bmatrix}.$$

Solution:

$$A^{-1}=\frac{1}{2(2-3i)-4}\begin{bmatrix} 2 & -4 \\ -1 & 2-3i \end{bmatrix}$$

$$=-\frac{1}{6i}\begin{bmatrix} 2 & -4 \\ -1 & 2-3i \end{bmatrix}$$

$$=\frac{i}{6}\begin{bmatrix} 2 & -4 \\ -1 & 2-3i \end{bmatrix}$$

$$=\begin{bmatrix} i/3 & -2i/3 \\ -i/6 & i/3+1/2 \end{bmatrix}=\begin{bmatrix} i/3 & -2i/3 \\ -i/6 & \frac{3+2i}{6} \end{bmatrix}$$

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#7

Show that 10 does not have a multiplicative inverse in $\mathbb{Z}_{15}$.

Solution:

Suppose for contradiction there is an element $a\in\mathbb{Z}_{15}$ such that

$$10\cdot a=1$$

$$\Rightarrow 30\cdot a=3$$

$$\Rightarrow 0=3$$

which is a contradiction. Consequently, 10 is not invertible in $\mathbb{Z}_{15}$.

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#8

Find all solutions to the equation $x^2=1$ in $\mathbb{Z}_8$.

Solution:

We can simply set up a table:

x	0	1	2	3	4	5	6	7
x^2	0	1	4	1	0	1	4	1

Consequently, in $\mathbb{Z}_8$ the solutions are

$$x = 1, 3, 5, 7$$

#9

Find a quadratic polynomial x^2+bx+c over $\mathbb{Z}_6$ which has four distinct roots in $\mathbb{Z}_6$.

Solution:

Lets consider $p(x)=x^2+bx+c$. Constructing a table we have that

x	0	1	2	3	4	5
$p(x)$	c	$1+b+c$	$4+2b+c$	$3+3b+c$	$4+4b+c$	$1+5b+c$

Now, suppose we want

$$b+c=0$$
$$4+2b+c=0$$
$$4+4b+c=0$$
$$1+5b+c=0$$

Therefore, $c=-b$ and we obtain

$$4+b=0 \quad (1)$$
$$4+3b=0 \quad (2)$$
$$1+4b=0 \quad (3)$$

By equation (1) this implies $b=2$ which by equation (2) implies $4=0$ which is a contradiction.

If instead we assume

$c = 0, b = 1$

we have:

x	0	1	2	3	4	5
$p(x)$	0	1	0	0	2	0

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#10

Find two nonzero elements in $\mathbb{Z}_{14}$ whose product is 0.

Solution:

7 and 2 work

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#11

Solve the following system of linear equations over $\mathbb{Z}_3$:

$$2x + y = 1$$
$$2x + 2y = 2$$

Solution:

$$\left[\begin{array}{cc:c} 2 & 1 & 1 \\ 2 & 2 & 2 \end{array}\right] \times 2 \rightarrow \left[\begin{array}{cc:c} 1 & 2 & 2 \\ 2 & 2 & 2 \end{array}\right] + R1 \rightarrow \left[\begin{array}{cc:c} 1 & 2 & 2 \\ 0 & 1 & 1 \end{array}\right] + R2 \rightarrow \left[\begin{array}{cc:c} 1 & 0 & 0 \\ 0 & 1 & 1 \end{array}\right]$$

Therefore, $y = 1$ and $x = 0$.

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#12

Find all solutions to the following system of linear equations over $\mathbb{Z}_5$:

$$2w+2x+y+z=4$$
$$w+x+3y+z=1$$
$$x+y=1$$

Solution:

$$\left[\begin{array}{cccc|c} 2 & 2 & 1 & 1 & 4 \\ 1 & 1 & 3 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{array}\right] \times 3 \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 3 & 3 & 2 \\ 1 & 1 & 3 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{array}\right] +4R1 \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 3 & 3 & 2 \\ 0 & 0 & 0 & 3 & 4 \\ 0 & 1 & 1 & 0 & 1 \end{array}\right] \times 2 \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 3 & 3 & 2 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right] +4R2$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right] +2R3 \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 2 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

Consequently

$$w+2y=2$$
$$x+y=1$$
$$z=3$$

$$\Rightarrow w=3y+2$$
$$x=1+4y$$
$$z=3.$$

Taking cases:

$y=0$	$y=1$	$y=2$	$y=3$	$y=4$
$w=2$	$w=0$	$w=3$	$w=1$	$w=4$
$x=1$	$x=0$	$x=4$	$x=3$	$x=2$
$z=3$	$z=3$	$z=3$	$z=3$	$z=3.$

#13

Find the inverse of the following matrix over $\mathbb{Z}_3$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Solution:

Row reducing, we have

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]\begin{array}{l} \\ +R1 \\ +2R1 \end{array} \to \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 2 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]\begin{array}{l} \\ \times 2 \\ \\ \end{array}$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]\begin{array}{l} +R2 \\ \\ \\ \end{array} \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]\begin{array}{l} +2R3 \\ \\ \\ \end{array} \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & 2 \\ 0 & 1 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

Consequently,

$$A^{-1} = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix}$$

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