MTH 225
Homework #3

Due Date: February 05, 2025

1. Let $Q = \{(x, y) : x, y > 0\} \subset \mathbb{R}^2$ denote the positive quadrant in the real plane.
   (a) Show that Q is not a vector space over $\mathbb{R}$ under the standard rules for vector addition and scalar multiplication.
   (b) Show that Q is a vector space over $\mathbb{R}$ if we define vector addition by
   $$(x_1, y_1) + (x_2, y_2) = (x_1 x_2, y_1 y_2)$$
   and scalar multiplication by
   $$c(x, y) = (x^c, y^c).$$
   Hint: You will have to show all of the properties of a vector space are true. You will also have to determine the zero vector $\mathbf{0}$.

2. Let $V \subset M_{2\times 2}(\mathbb{C})$ be the set of complex matrices of the form
   $$\mathbf{v} = \begin{bmatrix} 1 & v \\ 0 & 1 \end{bmatrix}$$
   for some $v \in \mathbb{C}$. Define vector addition and scalar multiplication over the field $F = \mathbb{C}$ by
   $$\mathbf{v} + \mathbf{w} = \begin{bmatrix} 1 & v \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & w \\ 0 & 1 \end{bmatrix} \text{ (ordinary matrix multiplication)}$$
   and
   $$c\mathbf{v} = \begin{bmatrix} 1 & cv \\ 0 & 1 \end{bmatrix}.$$
   Prove that V is a vector space over $\mathbb{C}$. **Hint**: You will have to show all of the properties of a vector space are true. You will also have to determine the zero vector $\mathbf{0}$.

2 3. Consider the vector space $V = \mathbb{R}^2$.
   1 (a) Construct an example of a subset $S \subset V$ in which for all $c \in \mathbb{R}$ and $\mathbf{u} \in S$, $c\mathbf{u} \in S$ but S is not a subspace of V.
   1 (b) Construct an example of a subset $S \subset V$ in which for all $\mathbf{u}, \mathbf{v} \in V$, $\mathbf{u} + \mathbf{v} \in S$ but S is not a subspace of V.

4. Determine which of the following sets of vectors $\mathbf{u} = (u_1, \ldots, u_n)^T$ are subspaces of $\mathbb{C}^n$. If a set is a subspace, prove it, if not, present a counterexample.

   (a) $A = \{\mathbf{u} \in \mathbb{C}^n : u_1 = \cdots = u_n\}$

   (b) $B = \{\mathbf{u} \in \mathbb{C}^n : \mathrm{Re}(u_i) \geq 0\}$

   (c) $C = \{\mathbf{u} \in \mathbb{C}^n : u_1 = u_n = 0\}$

   (d) $D = \{\mathbf{u} \in \mathbb{C}^n : u_1 - u_n = 1\}$

2 5. Determine which of the following are vector spaces. If a set with the given operations is a vector space, prove it, if not, provide a counterexample. **Hint:** You can show something is a vector space by proving it is a subspace of one of the known vector spaces.

   (a) $A = \{f \in C^1(\mathbb{R}) : f(x+1) = f(x)\}$ with the standard operations of addition and scalar multiplication of functions.

   (b) $B = \{f \in C^1(\mathbb{R}) : f(x) \geq 0\}$ with the standard operations of addition and scalar multiplication of functions.

   (c) $C = \{p \in P_n(\mathbb{R}) : p(x) = p(-x)$ with the standard operations of addition and scalar multiplication of polynomials.

   (d) $D = \{p \in P_\infty(\mathbb{R}) : p(x)$ has a factor of $x - 1\}$ with the standard operations of addition and scalar multiplication of polynomials.

2 6. Determine which of the following sets are subspaces of $C^1(\mathbb{R})$, the vector space of all differentiable functions with continuous derivatives. If a set is a subspace, prove it, if not, present a counterexample.

   ½ (a) $A = \{f \in C^1(\mathbb{R}) : f(2) = f(3)\}$

   (b) $B = \{f \in C^1(\mathbb{R}) : f'(2) = f(3)\}$

   ½ (c) $C = \{f \in C^1(\mathbb{R}) : f'(x) + f(x) = 0\}$

   ½ (d) $D = \{f \in C^1(\mathbb{R}) : f(2 - x) = f(x)\}$

   ½ (e) $E = \{f \in C^1(\mathbb{R}) : f(-x) = e^x f(x)\}$

   (f) $F = \{f \in C^1(\mathbb{R}) : f(x) = a + b|x|$ for some $a, b \in \mathbb{R}\}$

2 7. Prove that the set of functions $u(x)$ that satisfy the following differential equation

$$\frac{d^2u}{dx^2} = xu(x)$$

forms a subspace of $C^2(\mathbb{R})$.

2 8. Let $A \in M_{3\times3}(\mathbb{C})$. Determine which of the following subsets of $M_{3\times3}(\mathbb{C})$ are subspaces of $M_{3\times3}(\mathbb{C})$. If a set is a subspace, prove it, if not, present a counterexample.

   (a) $A = \{X \in M_{3\times3}(\mathbb{C}) : X$ is invertible$\}$.

   (b) $B = \{X \in M_{3\times3}(\mathbb{C}) : X$ is not invertible$\}$.

   (c) $C = \{X \in M_{3\times3}(\mathbb{C}) : X$ is lower triangular$\}$.

   (d) $D = \{X \in M_{3\times3}(\mathbb{C}) : AX + X^TA = 0\}$, where X^T denotes the transpose of a matrix.

Homework #3

#1

Let $Q=\{(x,y)\in\mathbb{R}^2 : x,y>0\}\subseteq\mathbb{R}^2$ denote the positive quadrant in $\mathbb{R}^2$.

(a) Show that Q is not a vector space over $\mathbb{R}$ under the standard rules for vector addition and scalar multiplication.

(b) Show that Q is a vector space over $\mathbb{R}$ if we define addition by

$$(x_1,y_1)+(x_2,y_2)=(x_1x_2,y_1y_2)$$

and scalar multiplication by

$$c(x,y)=(x^c,y^c).$$

Solution:

(a) Q is not a vector space under the normal operations since $(0,0)=\vec{0}\notin Q$.

(b) Let $\vec{u}=(u_1,u_2)$, $\vec{v}=(v_1,v_2)$, $\vec{w}=(w_1,w_2)\in Q$ and $a,b\in\mathbb{R}$. Therefore,

1. $\vec{u}+\vec{v}=(u_1v_1,u_2v_2)$. Since $u_1,v_1,u_2,v_2>0$ it follows that $u_1v_1>0$ and $u_2v_2>0$ and thus $\vec{u}+\vec{v}\in Q$

2. $\vec{u}+\vec{v}=(u_1v_1,u_2v_2)=(v_1u_1,v_2u_2)=\vec{v}+\vec{u}$

3. $$\begin{aligned}\vec{u}+(\vec{v}+\vec{w})&=\vec{u}+(v_1w_1,v_2w_2)\\&=(u_1v_1w_1,u_2v_2w_2)\\&=(u_1v_1,u_2,v_2)+\vec{w}\\&=(\vec{u}+\vec{v})+\vec{w}.\end{aligned}$$

4. Let $\vec{0}=(1,1)$. Therefore
$$\vec{0}+\vec{u}=(1\cdot u_1,1\cdot u_2)=(u_1,u_2)=\vec{u}$$

5. Let $-\vec{u}=(1/u_1,1/u_2)$. Therefore,
$$\vec{u}+(-\vec{u})=(u_1/u_1,u_2/u_2)=(1,1)=\vec{0}.$$

6. $a\vec{u}=(u_1^a,u_2^a)$. Since $u_1,u_2>0$ it follows that $u_1^a,u_2^a>0$ and thus $a\vec{u}\in Q$.

7. $a(\vec{u}+\vec{v}) = a(u_1v_1, u_2v_2) = ((u_1v_1)^a, (u_2v_2)^a) = (u_1^a v_1^a, u_2^a v_2^a) = (u_1^a, u_2^a) + (v_1^a, v_2^a)$

$\Rightarrow a(\vec{u}+\vec{v}) = a\vec{u} + a\vec{v}.$

8. $(a+b)\vec{u} = (u_1^{a+b}, u_2^{a+b}) = (u_1^a u_1^b, u_2^a u_2^b) = (u_1^a, u_2^a) + (u_1^b, u_2^b) = a\vec{u} + b\vec{u}.$

9. $a(b\vec{u}) = a(u_1^b, u_2^b) = ((u_1^b)^a, (u_2^b)^a) = (u_1^{ba}, u_2^{ba}) = ab\,\vec{u}.$

10. $1\vec{u} = (u_1^1, u_2^1) = (u_1, u_2) = \vec{u}.$

By items 1-10, Q is a vector space with these operations.

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#3.

Consider the vector space $V = \mathbb{R}^2$.

(a) Construct an example of a subset $S \subset V$ in which for all $c \in \mathbb{R}$ and $\vec{u} \in S$, $c\vec{u} \in S$ but S is not a subspace.

(b) Construct an example of a subset $S \subset V$ in which for all $\vec{u}, \vec{v} \in V$, $\vec{u}+\vec{v} \in S$, but S is not a subspace of V.

Solution:

(a) Let $S = \{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : x \cdot y \geq 0 \}$ and $\vec{u} \in S$. Therefore, there exists $u_1, u_2 \in \mathbb{R}$ such that $u_1 \cdot u_2 \geq 0$ and $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$. Consequently,

$$c\vec{u} = \begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix}.$$

Since $cu_1 \cdot cu_2 = c^2 u_1 u_2 > 0$ it follows that $c\vec{u} \in S$.

Now, $\begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \end{bmatrix} \in S$ but $\begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \notin S$. Consequently, S is not a subspace.

(b) Let $S = \{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : x \geq 0, y \geq 0 \}$ and $\vec{u}, \vec{v} \in S$. Therefore, there exists $u_1, u_2, v_1, v_2 \in \mathbb{R}$ such that $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and thus $\vec{u}+\vec{v} = \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \end{bmatrix}$. Since $u_1, u_2, v_1, v_2 \geq 0$ it follows that $u_1+v_1, u_2+v_2 \geq 0$ and therefore $\vec{u}+\vec{v} \in S$.

Now, $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \in S$ but $-1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \notin S$. Consequently, S is not a subspace.

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#4

Determine, which of the following sets of vectors $\vec{v} = (v_1, \dots, v_n)^T$ are subspaces of $\mathbb{C}^n$

(a) $A = \{\vec{v} \in \mathbb{C}^n : v_1 = \dots = v_n\}$

(b) $B = \{\vec{v} \in \mathbb{C}^n : Re(v_i) \geq 0\}$

(c) $C = \{\vec{v} \in \mathbb{C}^n : v_1 = v_n = 0\}$.

(d) $D = \{\vec{v} \in \mathbb{C}^n : v_1 - v_n = 1\}$.

Solution:

(a) Let $\vec{v}, \vec{V} \in A$ and $\lambda \in \mathbb{C}$. Therefore, $\vec{v} = (v, \dots, v)$ and $\vec{V} = (V, \dots, V)$ for some $v, V \in \mathbb{C}$. Consequently,

$$\vec{v} + \lambda \vec{V} = (v, \dots, v) + \lambda (V, \dots, V) = (v + \lambda V, \dots, v + \lambda V) \in A.$$

Therefore, A is a subspace of $\mathbb{V}$.

(b) B is not a subspace since $(1, \dots, 1) \in B$ but $-1 \cdot (1, \dots, 1) \notin B$.

(c) Let $\vec{v}, \vec{V} \in C$ and $\lambda \in \mathbb{C}$. Therefore, there exists $v_2, \dots, v_{n-1}, V_2, \dots, V_{n-1} \in \mathbb{C}$ such that $\vec{v} = (0, v_2, \dots, v_{n-1}, 0)$, $\vec{V} = (0, V_2, \dots, V_{n-1}, 0)$ and thus

$$\vec{v} + \lambda \vec{V} = (0, v_2, \dots, v_{n-1}, 0) + \lambda (0, V_2, \dots, V_{n-1}, 0) = (0, v_2 + \lambda V_2, \dots, v_{n-1} + \lambda V_{n-1}, 0) \in C.$$

Therefore, C is a subspace of $\mathbb{C}^n$.

(d) D is not a subspace since $\vec{0} \notin D$.

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#5

Determine which of the following are vector spaces.

(a) $A = \{f \in C^1(\mathbb{R}) : f(x+1) = f(x)\}$.

(b) $B = \{f \in C^1(\mathbb{R}) : f(x) \geq 0\}$

(c) $C = \{p \in P_n(\mathbb{R}) : p(x) = p(-x)\}$.

(d) $D = \{p \in P_\infty(\mathbb{R}) : p(x) \text{ has a factor of } x-1\}$.

Solution:

(a) Let $f, g \in A$ and $\lambda \in \mathbb{R}$. If $h = f + \lambda g$ then
$$h(x+1) = f(x+1) + \lambda g(x+1) = f(x) + \lambda g(x) = h(x).$$
Therefore, A is a subspace of $\mathcal{C}^1(\mathbb{R})$.

(b) B is not a subspace since $1 \in B$ but $-1 \cdot 1 = -1 \notin B$.

(c) Let $p, q \in C$ and $\lambda \in \mathbb{R}$. If $r = p + \lambda q$ then
$$r(x) = p(x) + \lambda q(x) = p(-x) + \lambda q(-x) = r(-x).$$
Therefore, $r \in C$ and thus C is a subspace of $P_n(\mathbb{R})$.

(d) Let $p, q \in D$ and $\lambda \in \mathbb{R}$. Therefore, there exists polynomials r, s such that
$$p(x) = (x-1)\, r(x),$$
$$q(x) = (x-1)\, s(x),$$
Let $t(x) = p(x) + \lambda q(x)$. Consequently,
$$t(x) = (x-1) r(x) + \lambda (x-1) s(x) = (x-1)(r(x) + \lambda s(x)).$$
and thus $t \in D$. Therefore, D is a subspace of $P_\infty(\mathbb{R})$.

#6.

Determine which of the following are subspaces of $C^1(\mathbb{R})$.

(a) $A = \{ f \in \mathcal{C}^1(\mathbb{R}) : f(2) = f(3) \}$

(c) $C = \{ f \in \mathcal{C}^1(\mathbb{R}) : f'(x) + f(x) = 0 \}$

(e) $E = \{ f \in \mathcal{C}^1(\mathbb{R}) : f(-x) = e^x f(x) \}$

(f) $F = \{ f \in \mathcal{C}^1(\mathbb{R}) : f(x) = a + b|x| \text{ for some } a, b \in \mathbb{R} \}$.

Solution:

(a) Let $f, g \in A$, $\lambda \in \mathbb{R}$ and $h = f + \lambda g$. Consequently
$$h(2) = f(2) + \lambda g(2) = f(3) + \lambda g(3) = h(3)$$
Therefore, A is a subspace of $C^1(\mathbb{R})$.

(c) Let $f, g \in C$, $\lambda \in \mathbb{R}$ and $h = f + \lambda g$. Consequently,
$$h'(x) + h(x) = f'(x) + \lambda g'(x) = 0 + \lambda \cdot 0 = 0.$$
Therefore, C is a subspace of $C^1(\mathbb{R})$.

(e) Let $f, g \in E$, $\lambda \in \mathbb{R}$ and $h = f + \lambda g$. Therefore,

$$h(-x) = f(-x) + \lambda g(-x) = e^x f(x) + \lambda e^x g(x) = e^x(f(x) + \lambda g(x)) = e^x h(x).$$

Therefore, E is a subspace of $C^1(\mathbb{R})$.

(f) Let $f, g \in F$, $\lambda \in \mathbb{R}$ and $h = f + \lambda g$. Therefore, there exists $a, b, c, d \in \mathbb{R}$ such that $f(x) = a + b|x|$ and $g(x) = c + d|x|$. Consequently,

$$h(x) = a + b|x| + c + d|x| = a + c + (b+d)|x|$$

and thus F is a subspace of $C^1(\mathbb{R})$.

#7

Prove that the set of functions that satisfy the following differential equation

$$\frac{d^2 v}{dx^2} = x v(x)$$

forms a subspace of $C^2(\mathbb{R}^n)$.

Solution:

Let v, V satisfy the differential equation and set $h(x) = v(x) + \lambda V(x)$. Therefore,

$$\frac{d^2 h}{dx^2} = \frac{d^2 v}{dx^2} + \lambda \frac{d^2 V}{dx^2} = x v(x) + \lambda x V(x) = x(v(x) + \lambda V(x)) = x h(x).$$

Consequently, the set of solutions to $\frac{d^2 v}{dx^2} = x v(x)$ forms a subspace of $C^2(\mathbb{R}^n)$.

#8

Let $A \in M_{3\times 3}(\mathbb{C})$. Determine which of the following subsets of $M_{3\times 3}(\mathbb{C})$ are subspaces of $M_{3\times 3}(\mathbb{C})$.

(a) $A = \{X \in M_{3\times 3}(\mathbb{C}) : X \text{ is invertible}\}$.

(b) $B = \{X \in M_{3\times 3}(\mathbb{C}) : X \text{ is not invertible}\}$.

(c) $C = \{X \in M_{3\times 3}(\mathbb{C}) : X \text{ is lower triangular}\}$.

(d) $D = \{X \in M_{3\times 3}(\mathbb{C}) : AX + X^T A = 0\}$.

Solution:

(a) $0 \notin A$ and thus A is not a subspace of $M_{3\times 3}(\mathbb{R})$.

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in B$ and $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in B$ but $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \notin B$

and thus B is not a subspace of $M_{2\times 2}(\mathbb{R})$.

(c) Let $A, B \in C$. Therefore, there exists $a_1, a_2, a_3, a_4, a_5, a_6 \in \mathbb{R}$ and $b_1, b_2, b_3, b_4, b_5, b_6 \in \mathbb{R}$ such that

$$A = \begin{bmatrix} a_1 & 0 & 0 \\ a_4 & a_2 & 0 \\ a_6 & a_5 & a_3 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 & 0 & 0 \\ b_4 & b_2 & 0 \\ b_6 & b_5 & b_6 \end{bmatrix}.$$

Consequently

$$A + \lambda B = \begin{bmatrix} a_1+\lambda b_1 & 0 & 0 \\ a_4+\lambda b_4 & a_2+\lambda b_2 & 0 \\ a_6+\lambda b_6 & a_5+\lambda b_5 & a_3+\lambda b_6 \end{bmatrix} \in C.$$

Therefore, C is a subspace of $M_{2\times 2}(\mathbb{R})$.

(d) Let $X, Y \in D$, $\lambda \in \mathbb{R}$, and $Z_1 = X + \lambda Y$. Therefore,

$$\begin{aligned} AZ_1 + Z_1^T A &= A(X+\lambda Y) + (X+\lambda Y)^T A \\ &= AX + \lambda AY + (X^T + \lambda Y^T)A \\ &= AX + \lambda AY + X^T A + \lambda Y^T A \\ &= AX + X^T A + \lambda(AY + Y^T A) \\ &= 0 + \lambda 0 \\ &= 0. \end{aligned}$$

Consequently $Z_1 \in D$ and thus D is a subspace of $M_{2\times 2}(\mathbb{R})$.