MTH 225
Homework #4

Due Date: February 12, 2025

1. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}$ be vectors in a vector space V. Prove or provide a counterexample: if $\mathbf{z}$ is a linear combination of $\mathbf{u}, \mathbf{v}, \mathbf{w}$ then $\mathbf{w}$ is a linear combination of $\mathbf{u}, \mathbf{v}, \mathbf{z}$.

2. Suppose $\mathbf{f}(t)$ and $\mathbf{g}(t)$ are functions from $\mathbb{R}$ to $\mathbb{R}^2$.

   (a) Prove that if $\mathbf{f}(t_0)$ and $\mathbf{g}(t_0)$ are linearly independent as vectors in $\mathbb{R}^2$ for some $t_0 \in \mathbb{R}^2$ then $\mathbf{f}(t)$ and $\mathbf{g}(t)$ are linearly independent functions.

   (b) Show that $\mathbf{f}(t) = \begin{bmatrix} 1 \\ t \end{bmatrix}$ and $\mathbf{g}(t) = \begin{bmatrix} 2t - 1 \\ 2t^2 - t \end{bmatrix}$ are linearly independent functions.

   (c) Show that $\mathbf{f}(t_0) = \begin{bmatrix} 1 \\ t_0 \end{bmatrix}$ and $\mathbf{g}(t_0) = \begin{bmatrix} 2t_0 - 1 \\ 2t_0^2 - t_0 \end{bmatrix}$ are linearly dependent for all $t_0 \in \mathbb{R}$.

   Parts (b) and (c) show that the converse of (a) is not true in general.

3. The Wronskian of a pair of differentiable functions $f(x)$, $g(x)$ defined by

$$W[f(x), g(x)] = \det\left(\begin{bmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{bmatrix}\right).$$

   (a) Prove that if f and g are linearly dependent then $W[f(x), g(x)] = 0$.

   (b) Use part (a) to prove that if $W[f(x), g(x)] \neq 0$ then f and g are linearly independent.

   (c) Let $f(x) = x^3$ and $g(x) = |x|^3$. Show that f and g are linearly independent but $W[f(x), g(x)] = 0$. Why does this not contradict parts (a) and (b)?

4. Consider the set $\mathcal{A} = \{1 + x^2, x + x^2, 1 + 2x + x^2\}$.

   (a) Prove that $\mathcal{A}$ is a basis for $P_2(\mathbb{R})$.

   (b) Find the coordinates of $p(x) = 1 + 4x + 7x^2$ in this basis.

5. Let $S_{n\times n}(\mathbb{R})$ denote the subspace of $M_{2\times 2}(\mathbb{R})$ consisting of all symmetric $n \times n$ matrices with complex entries.

   (a) Show that a basis for $S_{2\times 2}(\mathbb{R})$ is given by

$$\mathcal{A} = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right\}.$$

   (b) Find a basis for $S_{3\times 3}(\mathbb{R})$ and determine $\dim(S_{3\times 3}(\mathbb{R}))$. You do not have to prove that your choice is a basis.

6. A real valued matrix is said to be a *semi-magic square* if its row sums and column sums, i.e, the sum of entries in an individual row or column, all add up to the sum number. An example is

$$M = \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix}.$$

   (a) Prove that S, the set of all 3×3 semi-magic squares, is a subspace of $M_{3\times3}(\mathbb{R})$.

   (b) Find a basis for S. You do not have to prove that your choice is a basis and, as a hint, $\dim(S) = 6$.

7. Find a linear transformation $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$ such that

$$T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \text{ and } T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

8. Consider the map $T : P_3(\mathbb{R}) \mapsto \mathbb{R}^4$ defined by

$$T(a_3x^3 + a_2x^2 + a_1 + a_0) = \begin{bmatrix} a_1 \\ -a_1 \\ a_2 \\ -a_2 \end{bmatrix}.$$

   (a) Show that T is a linear transformation.

   (b) Find bases for $\ker(T)$ and $\operatorname{im}(T)$.

9. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ be 2×2 real valued matrices. For each of the following functions, prove that $T : M_{2\times2}(\mathbb{R}) \mapsto M_{2\times2}(\mathbb{R})$ is a linear transformation and then find its 4×4 matrix representation with respect to the standard basis:

$$\mathcal{A} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

   (a) $T[X] = AX$.

   (b) $R[X] = XB$.

   (c) $K[X] = AXB$.

Homework #4

#1

Let $\vec{u}, \vec{v}, \vec{w}, \vec{z}$ be vectors in a vector space V. Prove or provide a counterexample: If $\vec{z}$ is a linear combination of $\vec{u}, \vec{v}, \vec{w}, \vec{z}$ then $\vec{w}$ is a linear combination of $\vec{u}, \vec{v}, \vec{z}$.

Solution:

Counterexample: Let

$$\vec{u} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \vec{v} = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \vec{w} = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}, \vec{z} = \begin{bmatrix}1\\1\\0\\0\end{bmatrix}$$

Consequently, $\vec{z} = 1\cdot\vec{u} + 1\vec{v} + 0\cdot\vec{w}$ but $\vec{w} \notin \text{span}\{\vec{u}, \vec{v}, \vec{z}\}$.

#2

Suppose $\vec{f}, \vec{g}$ are functions from $\mathbb{R} \to \mathbb{R}^2$

(a) Prove that if $\vec{f}(t_0)$ and $\vec{g}(t_0)$ are linearly independent as vectors in $\mathbb{R}^2$ for some $t_0 \in \mathbb{R}$ then $\vec{f}(t), \vec{g}(t)$ are linearly independent functions.

(b) Show that $\vec{f}(t) = \begin{bmatrix}1\\t\end{bmatrix}$ and $\vec{g}(t) = \begin{bmatrix}2t-1\\2t^2-t\end{bmatrix}$ are linearly independent functions.

(c) Show that $\vec{f}(t_0) = \begin{bmatrix}1\\t_0\end{bmatrix}$ and $\vec{g}(t_0) = \begin{bmatrix}2t_0-1\\2t_0^2-t_0\end{bmatrix}$ are linearly dependent for all $t_0 \in \mathbb{R}$

Solution:

(a) Suppose for some $t_0 \in \mathbb{R}$, the only solution to the equation $c_1\vec{f}(t_0) + c_2\vec{g}(t_0) = \vec{0}$ is $c_1 = c_2 = 0$. Therefore, for all t, the only solution to the equation

$$c_1\vec{f}(t) + c_2\vec{g}(t) = \vec{0}$$

is $c_1 = c_2 = 0$ which implies that $\vec{f}, \vec{g}$ are linearly ind. as functions.

(b) Now, consider the equation

$$c_1\begin{bmatrix}1\\t\end{bmatrix}+c_2\begin{bmatrix}2t-1\\2t^2-t\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

Setting $t=0$ we obtain

$$c_1-c_2=0 \Rightarrow c_1=c_2 \quad (*)$$

Setting $t=1$ we obtain

$$c_1\begin{bmatrix}1\\1\end{bmatrix}+c_2\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

$$\Rightarrow c_1=-c_2 \quad (**)$$

Therefore, the only combination that satisfies $(*)$ and $(**)$ is given by $c_1=c_2=0$ and thus $\vec{f},\vec{g}$ are linearly independent.

(c) Evaluating at t_0, we have the equation

$$c_1\begin{bmatrix}1\\t_0\end{bmatrix}+c_2\begin{bmatrix}2t_0-1\\2t_0^2-t_0\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}1 & 2t_0-1\\t_0 & t_0(2t_0-1)\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

Now

$$\det\left(\begin{bmatrix}1 & 2t_0-1\\t_0 & t_0(2t_0-1)\end{bmatrix}\right)=t_0(2t_0-1)-t_0(2t_0-1)=0.$$

Thus, as vectors in $\mathbb{R}^2$, $\vec{f}(t_0),\vec{g}(t_0)$ are linearly dependent. ∎

#3

The Wronskian of a pair of differentiable functions $f(x), g(x)$ is defined by

$$W[f(x),g(x)] = \det\left(\begin{bmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{bmatrix}\right)$$

(a) Prove that if f, g are linearly dependent then

$$W[f(x), g(x)] = 0.$$

(b) Use part (a) to prove that if $W[f(x),g(x)] \neq 0$ then f and g are linearly independent.

(c) Let $f(x) = x^3$ and $g(x) = |x|^3$. Show that f and g are linearly independent but $W[f(x), g(x)] = 0$.

Solution:

(a) If f, g are linearly dependent then there exists $c \in \mathbb{R}$ such that

$$f(x) = cg(x).$$

Differentiating, we have

$$f'(x) = cg'(x).$$

Consequently,

$$\det\left(\begin{bmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{bmatrix}\right) = \det\left(\begin{bmatrix} cg(x) & g(x) \\ cg'(x) & g'(x) \end{bmatrix}\right)$$
$$= cg(x)g'(x) - cg(x)g'(x)$$
$$= 0.$$

(b) If we assume $W[f(x),g(x)] \neq 0$ and f, g are linearly dependent then we obtain the contradiction $W[f(x),g(x)] = 0$. Consequently, f, g are linearly independent.

(c) If we consider the equation

$$c_1 f(x) + c_2 g(x) = 0$$

and set $x=1$ and $x=-1$ we obtain

$$c_1 + c_2 = 0$$
$$-c_1 + c_2 = 0$$

which implies $c_1 = c_2 = 0$. Therefore, f, g are linearly independent.

Now,

$$g(x) = \begin{cases} x^3, & x \geq 0 \\ -x^3, & x < 0 \end{cases} \Rightarrow g'(x) = \begin{cases} 3x^2, & x \geq 0 \\ -3x^2, & x \leq 0 \end{cases}$$

Therefore,

$$\det(W[f,g]) = \begin{cases} \det\left(\begin{bmatrix} x^3 & x^3 \\ 3x^2 & 3x^2 \end{bmatrix}\right), & x \geq 0 \\ \det\left(\begin{bmatrix} x^3 & -x^3 \\ 3x^2 & -3x^2 \end{bmatrix}\right), & x < 0 \end{cases}$$
$$= 0.$$

#4.

Consider the set $A = \{1+x^2, x+x^2, 1+2x+x^2\}$.

(a) Prove that A is a basis for $P_2(\mathbb{R})$.

(b) Find the coordinates of $p(x) = 1 + 4x + 7x^2$ in this basis.

Solution:

(a) Let $p(x) = a + bx + cx^2$ for some $a, b, c \in \mathbb{R}$ and suppose

$$a + bx + cx^2 = c_1(1+x^2) + c_2(x+x^2) + c_3(1+2x+x^2)$$

for some $c_1, c_2, c_3 \in \mathbb{R}$

$$\Rightarrow a + bx + cx^2 = (c_1 + c_3) + (c_2 + 2c_3)x + (c_1 + c_2 + c_3)x^2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 1 & 2 & b \\ 1 & 1 & 1 & c \end{array}\right] \begin{matrix} \\ \\ -R1 \end{matrix} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 1 & 2 & b \\ 0 & 1 & 0 & c-a \end{array}\right] \begin{matrix} \\ \\ -R2 \end{matrix} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 1 & 2 & b \\ 0 & 0 & -2 & c-a-b \end{array}\right] \begin{matrix} \\ \\ /{-2} \end{matrix} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 1 & 2 & b \\ 0 & 0 & 1 & (a+b-c)/2 \end{array}\right] \begin{matrix} -R3 \\ -2 \cdot R3 \\ \\ \end{matrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & (a-b+c)/2 \\ 0 & 1 & 0 & (c-a) \\ 0 & 0 & 1 & (a+b-c)/2 \end{array}\right]$$

Consequently, the unique solution is

$$c_1 = (a-b+c)/2$$
$$c_2 = c-a$$
$$c_3 = (a+b-c)/2$$

and thus A is a basis.

(b) For $p(x) = 1+4x+7x^2$, we have $a=1, b=4, c=7$ and thus

$$c_1 = 2$$
$$c_2 = 6$$
$$c_3 = -1$$

Consequently,

$$[p(x)]_A = \begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix}.$$

#5

Let $S_{n\times n}(\mathbb{R})$ denote the subspace of $M_{2\times 2}(\mathbb{R})$ consisting of all symmetric $n\times n$ matrices with complex entries.

(a) Show that a basis for $S_{2\times 2}(\mathbb{R})$ is given by

$$A = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}.$$

(b) Find a basis for $S_{3\times 3}(\mathbb{R})$ and determine $\dim(S_{3\times 3}(\mathbb{R}))$.

Solution:

(a) If $A \in S_{2\times 2}(\mathbb{R})$ then there exists $a, b, c \in \mathbb{R}$ such that

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

and thus $\mathcal{A}$ is a basis.

(b) If $B \in S_{3\times 3}(\mathbb{R})$ then there exists $a, b, c, d, e, f \in \mathbb{R}$ such that

$$B = \begin{bmatrix} a & d & f \\ d & b & e \\ f & e & c \end{bmatrix} = a\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + d\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + e\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + f\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and thus a basis is given by

$$\beta = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right\}$$

$\Rightarrow \dim(S_{3\times 3}(\mathbb{R})) = 6.$

#6

(a) Prove that S, the set of 3x3 semi-magic squares, is a subspace of $M_{3\times 3}(\mathbb{R})$.

(b) Find a basis for S.

Solution:

(a) Let $A, B \in S$ and $\lambda \in \mathbb{R}$. Therefore, there exists $a, b \in \mathbb{R}$ such that the rows and columns of A, B sum to a, b. Consequently, if we let $C = A + \lambda B$ then the row and column sums are given by $a + \lambda b$ and thus $C \in S$. Therefore, S is a subspace.

(b) The set

$$\mathcal{A} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right\}$$

is a basis and thus

$\dim(S) = 6.$

#7

Find a linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$T\left(\begin{bmatrix}1\\2\end{bmatrix}\right)=\begin{bmatrix}2\\-1\end{bmatrix},\ T\left(\begin{bmatrix}2\\1\end{bmatrix}\right)=\begin{bmatrix}0\\-1\end{bmatrix}.$$

Solution:

Let T have the representation

$$T=\begin{bmatrix}a & b\\c & d\end{bmatrix}.$$

Consequently,

$$\begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}2\\-1\end{bmatrix} \text{ and } \begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}0\\-1\end{bmatrix}$$

$$\Rightarrow \begin{array}{l} a+2b=2 \\ c+2d=-1 \end{array} \quad \text{and} \quad \begin{array}{l} 2a+b=0 \\ 2c+d=-1 \end{array}$$

$$\Rightarrow \begin{bmatrix}1 & 2\\2 & 1\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}2\\0\end{bmatrix} \text{ and } \begin{bmatrix}1 & 2\\2 & 1\end{bmatrix}\begin{bmatrix}c\\d\end{bmatrix}=\begin{bmatrix}-1\\-1\end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{cc|c}1 & 2 & 2\\2 & 1 & 0\end{array}\right]-2R1 \qquad \left[\begin{array}{cc|c}1 & 2 & -1\\2 & 1 & -1\end{array}\right]-2R1$$

$$\Rightarrow \left[\begin{array}{cc|c}1 & 2 & 2\\0 & -3 & -4\end{array}\right]/-3 \qquad \left[\begin{array}{cc|c}1 & 2 & -1\\0 & -3 & 1\end{array}\right]/-3$$

$$\Rightarrow \left[\begin{array}{cc|c}1 & 2 & 2\\0 & 1 & 4/3\end{array}\right]\begin{array}{l}-2R2\\ \\ \end{array} \qquad \left[\begin{array}{cc|c}1 & 2 & -1\\0 & 1 & -1/3\end{array}\right]\begin{array}{l}-2R2\\ \\ \end{array}$$

$$\Rightarrow \left[\begin{array}{cc|c}1 & 0 & -2/3\\0 & 1 & 4/3\end{array}\right] \qquad \left[\begin{array}{cc|c}1 & 0 & -1/3\\0 & 1 & -1/3\end{array}\right]$$

Consequently,

$a=-2/3$

$b=4/3$

$c=-1/3$

$d=-1/3$

and therefore

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -2/3 & 4/3 \\ -1/3 & -1/3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{2}{3}x + \frac{4}{3}y \\ -\frac{1}{3}x - \frac{1}{3}y \end{bmatrix}$$

$\blacksquare$

#8

Consider the map $T: P_3(\mathbb{R}) \to \mathbb{R}^4$ by

$$T(a_3x^3 + a_2x^2 + a_1x + a_0) = \begin{bmatrix} a_1 \\ -a_1 \\ a_2 \\ -a_2 \end{bmatrix}$$

(a) Show that T is a linear transformation.

(b) Find bases for $\ker(T)$ and $\mathrm{im}(T)$.

Solution:

(a) Let $p_1(x), p_2(x) \in P_3(\mathbb{R})$. Therefore, there exists $a_3, a_2, a_1, a_0, b_3, b_2, b_1, b_0 \in \mathbb{R}$ such that

$$p_1(x) = a_3x^3 + a_2x^2 + a_1x + a_0,$$
$$p_2(x) = b_3x^3 + b_2x^2 + b_1x + b_0.$$

Consequently, for $\lambda \in \mathbb{R}$ it follows that

$$T(p_1(x) + \lambda p_2(x)) = \begin{bmatrix} a_1 + \lambda b_1 \\ -a_1 - \lambda b_1 \\ a_2 + \lambda b_2 \\ -a_2 - \lambda b_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ -a_1 \\ a_2 \\ -a_2 \end{bmatrix} + \lambda \begin{bmatrix} b_1 \\ -b_1 \\ b_2 \\ -b_2 \end{bmatrix} = T(p_1) + \lambda(p_2).$$

Therefore, T is a linear transformation.

(b) $\ker(T) = \mathrm{span}\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}$

$\mathrm{im}(T) = \mathrm{span}\left\{\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}\right\}$

$\blacksquare$

#9

Let $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B=\begin{bmatrix} p & q \\ r & s \end{bmatrix}$ be real 2x2 matrices. For each of the following functions, prove that $T: M_{2\times2}(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$ is a linear transformation and then find its 4x4 matrix representation with respect to the standard basis.

(a) $T[X]=AX$

(b) $R[X]=XB$

(c) $K[X]=AXB$

Solution:

(a) Let $X, Y \in M_{2\times2}(\mathbb{R})$ and $\lambda \in \mathbb{R}$. Therefore,

$$T[X+\lambda Y]=A(X+\lambda Y)=AX+\lambda AY=T[X]+\lambda T[Y]$$

and thus T is linear.

Now,

$$T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}=\begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix}=a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}+c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}=\begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix}=a\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}+c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right)=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}=\begin{bmatrix} b & 0 \\ d & 0 \end{bmatrix}=b\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}+d\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}=\begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix}=b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}+d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore, the matrix representation is given by:

$$\begin{bmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{bmatrix}$$

(b) Let $X, Y \in M_{2\times2}(\mathbb{R})$ and $\lambda \in \mathbb{R}$. Therefore,

$$R[X+\lambda Y]=(X+\lambda Y)B=XB+\lambda YB=R[X]+\lambda R[Y].$$

and thus T is linear.

Now,

$$R\left(\begin{bmatrix}1&0\\0&0\end{bmatrix}\right)=\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}p&q\\r&s\end{bmatrix}=\begin{bmatrix}p&q\\0&0\end{bmatrix}=p\begin{bmatrix}1&0\\0&0\end{bmatrix}+q\begin{bmatrix}0&1\\0&0\end{bmatrix}$$
$$R\left(\begin{bmatrix}0&1\\0&0\end{bmatrix}\right)=\begin{bmatrix}0&1\\0&0\end{bmatrix}\begin{bmatrix}p&q\\r&s\end{bmatrix}=\begin{bmatrix}r&s\\0&0\end{bmatrix}=r\begin{bmatrix}1&0\\0&0\end{bmatrix}+s\begin{bmatrix}0&1\\0&0\end{bmatrix}$$
$$R\left(\begin{bmatrix}0&0\\1&0\end{bmatrix}\right)=\begin{bmatrix}0&0\\1&0\end{bmatrix}\begin{bmatrix}p&q\\r&s\end{bmatrix}=\begin{bmatrix}0&0\\p&q\end{bmatrix}=p\begin{bmatrix}0&0\\1&0\end{bmatrix}+q\begin{bmatrix}0&0\\0&1\end{bmatrix}$$
$$R\left(\begin{bmatrix}0&0\\0&1\end{bmatrix}\right)=\begin{bmatrix}0&0\\0&1\end{bmatrix}\begin{bmatrix}p&q\\r&s\end{bmatrix}=\begin{bmatrix}0&0\\r&s\end{bmatrix}=r\begin{bmatrix}0&0\\1&0\end{bmatrix}+s\begin{bmatrix}0&0\\0&1\end{bmatrix}$$

Therefore, the matrix representation is given by

$$\begin{bmatrix}p&r&0&0\\q&s&0&0\\0&0&p&r\\0&0&q&s\end{bmatrix}$$

(c) Let $A, B \in M_{2\times2}(\mathbb{R})$ and $\lambda \in \mathbb{R}$. Therefore,

$$K[X+\lambda Y]=A(X+\lambda Y)B=(AX+\lambda AY)B=AXB+\lambda AYB=K[X]+\lambda K[Y]$$

and thus K is a linear transformation.

To find the matrix representation we can commute

$$\begin{bmatrix}a&0&b&0\\0&a&0&b\\c&0&d&0\\0&c&0&d\end{bmatrix}\begin{bmatrix}p&r&0&0\\q&s&0&0\\0&0&p&r\\0&0&q&s\end{bmatrix}=\begin{bmatrix}ap&ar&bp&br\\aq&as&bq&bs\\cp&cr&dp&dr\\cq&cs&dq&ds\end{bmatrix}$$