MTH 225
Homework #5

Due Date: February 26, 2025

1. The set of vectors $\mathcal{A} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ and $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$ are a basis for $\mathbb{R}^2$.
   (a) Compute $[I(\mathcal{A}, \mathcal{B})]$ and $[I(\mathcal{B}, \mathcal{A})]$.
   (b) Show that $[I(\mathcal{A}, \mathcal{B})]$ and $[I(\mathcal{B}, \mathcal{A})]$ are inverses of each other.

2. Let $V = M_{2\times 2}(\mathbb{C})$ and $W = M_{n\times n}(\mathbb{C})$.
   (a) Find a standard basis for V. You don't need to prove anything, you can just list the matrices. **Hint:** $\dim(V) = 8$.
   (b) What is $\dim(W)$?
   (c) Let $\mathrm{RE} : V \mapsto M_{2\times 2}(\mathbb{R})$ be defined by $\mathrm{RE}(A)$ is the real valued matrix with components $(\mathrm{RE}(A))_{ij} = \mathrm{Re}(A_{ij})$. Show that RE is a linear transformation.
   (d) Find the matrix representation of RE with respect to the standard basis for V and the standard basis for $M_{2\times 2}(\mathbb{R})$.
   (e) Find a basis for $\ker(\mathrm{RE})$ and $\mathrm{im}(A)$. **Hint:** You don't have to do any row reduction or set up any system of linear equations.

3. For $A \in M_{n\times n}(\mathbb{C})$, with entries $A_{ij} \in \mathbb{C}$, the trace function $\mathrm{tr} : M_{n\times n}(\mathbb{C}) \mapsto \mathbb{C}$ is defined by
$$\mathrm{tr}(A) = \sum_{i=1}^{n} A_{ii}.$$
   (a) Show that tr is a linear transformation.
   (b) Explain why $\dim(\mathrm{im}(\mathrm{tr})) = 2$.
   (c) Determine $\dim(\ker(\mathrm{tr}))$.
   (d) Show that if $A, B \in M_{n\times n}(\mathbb{C})$ then $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. **Hint:** Remember that if $C = AB$, then the entries of C are given by $C_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}$.

4. Let $A, B \in M_{n\times n}(\mathbb{C})$. Recall that A and B are similar if there exists a matrix $P \in M_{n\times n}(\mathbb{C})$ such that $A = PBP^{-1}$.
   (a) Show that if A is similar to B, then $A - \lambda I$ is similar to $B - \lambda I$ for every $\lambda \in \mathbb{C}$.
   (b) Show that if there exists a $\lambda \in \mathbb{C}$ such that $A - \lambda I$ is similar to $B - \lambda I$, then A is similar to B.
   (c) Show that if A is similar to B then $\mathrm{tr}(A) = \mathrm{tr}(B)$.
   (d) Show that if A is similar to B then $\det(A) = \det B$.

5. Recall that a relationship on a set X, denoted $\sim$, is called an equivalence relationship if the following properties are satisfied

   (a) **Reflexivity:** If $a \in X$ then $a \sim a$.

   (b) **Symmetry:** If $a, b \in X$ then $a \sim b$ if and only if $b \sim a$.

   (c) **Transitivity:** If $a, b, c \in X$ and $a \sim b$ and $b \sim c$ then $a \sim c$.

   Show that on $M_{n \times n}(\mathbb{C})$ similarity between matrices is an equivalence relationship.

6. Let $A \in M_{n \times n}(\mathbb{C})$.

   (a) Show that A is similar to I if and only if $A = I$.

   (b) Show that A is similar to 0 if and only if $A = 0$.

7. Let $A \in M_{n \times n}(\mathbb{C})$.

   (a) Show that the linear system $A\mathbf{x} = \mathbf{y}$ has a unique solution for some $\mathbf{y} \in \mathbb{C}^n$ if and only if it has a solution for all $\mathbf{y} \in \mathbb{C}^n$.

   (b) Show that the linear system $A\mathbf{x} = \mathbf{y}$ has a unique solution for all $\mathbf{y} \in \mathbb{C}^n$ if and only if the only solution to $A\mathbf{x} = 0$ is $\mathbf{x} = 0$.

Homework #5

#1

The set of vectors $\mathcal{A}=\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\-1\end{bmatrix}\right\}$ and $\beta=\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}1\\3\end{bmatrix}\right\}$ are a basis for $\mathbb{R}^2$.

(a) Compute $[I(\mathcal{A},\beta)]$ and $[I(\beta,\mathcal{A})]$.

(b) Show that $[I(\mathcal{A},\beta)]$ and $[I(\beta,\mathcal{A})]$ are inverses of each other.

Solution:

(a) We need to convert $\mathcal{A}$ to the $\mathcal{B}$ basis. Assume

$$c_1\begin{bmatrix}1\\2\end{bmatrix}+c_2\begin{bmatrix}1\\3\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\-1\end{bmatrix}$$

$$\Rightarrow\left[\begin{array}{cc|cc}1&1&1&1\\2&3&1&-1\end{array}\right]\begin{array}{l}\\-2R\end{array}\rightarrow\left[\begin{array}{cc|cc}1&1&1&1\\0&1&-1&-3\end{array}\right]\begin{array}{l}-R2\\ \\ \end{array}\rightarrow\left[\begin{array}{cc|cc}1&0&2&4\\0&1&-1&-3\end{array}\right]$$

Therefore,

$$[I(\mathcal{A},\beta)]=\left[\begin{bmatrix}1\\1\end{bmatrix}_\beta\;\vdots\;\begin{bmatrix}1\\-1\end{bmatrix}_\beta\right]=\begin{bmatrix}2&4\\-1&-3\end{bmatrix}.$$

We also need to convert $\mathcal{B}$ to the $\mathcal{A}$ basis. Assume

$$c_1\begin{bmatrix}1\\1\end{bmatrix}+c_2\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}1\\3\end{bmatrix}$$

$$\Rightarrow\left[\begin{array}{cc|cc}1&1&1&1\\1&-1&2&3\end{array}\right]\begin{array}{l}\\-R1\end{array}\rightarrow\left[\begin{array}{cc|cc}1&1&1&1\\0&-2&1&2\end{array}\right]\begin{array}{l}\\/-2\end{array}\rightarrow\left[\begin{array}{cc|cc}1&1&1&1\\0&1&-\frac{1}{2}&-1\end{array}\right]\begin{array}{l}-R2\\ \\ \end{array}\rightarrow\left[\begin{array}{cc|cc}1&0&\frac{3}{2}&2\\0&1&-\frac{1}{2}&-1\end{array}\right]$$

Therefore,

$$[I(\beta,\mathcal{A})]=\left[\begin{bmatrix}1\\2\end{bmatrix}_\mathcal{A}\;\vdots\;\begin{bmatrix}1\\3\end{bmatrix}_\mathcal{A}\right]=\begin{bmatrix}\frac{3}{2}&2\\-\frac{1}{2}&-1\end{bmatrix}$$

Computing, we have that

$$[I(\beta,\mathcal{A})][I(\mathcal{A},\beta)]=\begin{bmatrix}\frac{3}{2}&2\\-\frac{1}{2}&-1\end{bmatrix}\begin{bmatrix}2&4\\-1&-3\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

and thus

$$[I(\beta,\mathcal{A})]^{-1}=[I(\mathcal{A},\beta)].$$ ■

#3

For $A \in M_{n\times n}(\mathbb{C})$, with entries $A_{ij} \in \mathbb{C}$, the trace function $tr: M_{n\times n}(\mathbb{C}) \to \mathbb{C}$ is defined by

$$tr(A) = \sum_{i=1}^{n} A_{ii}$$

(a) Show that tr is a linear transformation.

(b) Explain why $\dim(\mathrm{im}(tr)) = 2$.

(c) Determine $\dim(\ker(tr))$.

(d) Show that if $A, B \in M_{n\times n}(\mathbb{C})$ then $tr(AB) = tr(BA)$.

Solution:

(a) Let $A, B \in M_{n\times n}(\mathbb{C})$ and $\lambda \in \mathbb{C}$. Therefore,

$$tr(A+\lambda B) = \sum_{i=1}^{n}(A_{ii} + \lambda B_{ii}) = \sum_{i=1}^{n} A_{ii} + \lambda \sum_{i=1}^{n} B_{ii} = tr(A) + \lambda tr(B).$$

(b) Since tr can map to any complex number it follows that

$$\dim(\mathrm{im}(tr)) = \dim(\mathbb{C}) = 2.$$

(c) From the rank nullity theorem it follows that

$$\dim(\ker(tr)) + 2 = 2n$$

$$\Rightarrow \dim(\ker(tr)) = 2n - 2.$$

(d) Computing, we have that

$$tr(AB) = \sum_{i=1}^{n}\sum_{k=1}^{n} A_{ik} B_{ki} = \sum_{i=1}^{n}\sum_{k=1}^{n} B_{ki} A_{ik} = \sum_{k=1}^{n}\sum_{i=1}^{n} B_{ki} A_{ik} = tr(BA). \blacksquare$$

#4.

Let $A, B \in M_{n\times n}(\mathbb{C})$. Recall that A and B are similar matrices if there exists a matrix $P \in M_{n\times n}(\mathbb{C})$ such that $A = PBP^{-1}$.

(a) Show that if A is similar to B then $A-\lambda I$ is similar to $B-\lambda I$ for every $\lambda \in \mathbb{C}$.

(b) Show that if there exists a $\lambda \in \mathbb{C}$ such that $A-\lambda I$ is similar to $B-\lambda I$, then A is similar to B.

(c) Show that if A is similar to B then $tr(A) = tr(B)$.

(d) Show that if A is similar to B then $det(A) = det(B)$.

Solution:

(a) $A-\lambda I = PBP^{-1} - \lambda I = PBP^{-1} - \lambda PP^{-1} = P(B-\lambda I)P^{-1}$.

(b) If $A-\lambda I = P(B-\lambda I)P^{-1}$ it follows that

$$\begin{aligned} A &= P(B-\lambda I)P^{-1} + \lambda I \\ &= PBP^{-1} - \lambda PP^{-1} + \lambda I \\ &= PBP^{-1} - \lambda I + \lambda I \\ &= PBP^{-1}. \end{aligned}$$

(c)
$$\begin{aligned} tr(A) &= tr(PBP^{-1}) \\ &= tr(P^{-1}PB) \\ &= tr(B) \end{aligned}$$

(d)
$$\begin{aligned} det(A) &= det(PBP^{-1}) \\ &= det(P)det(B)det(P^{-1}) \\ &= \frac{det(P)\,det(B)}{det(P)} \\ &= det(B) \end{aligned}$$

■

#5

Show that on $M_{2\times2}(\mathbb{C})$ similarity between matrices is an equivalence relationship.

Solution:

(i) If $A \in M_{2\times2}(\mathbb{C})$ then $A = I^{-1}AI$ and thus $A \sim A$.

(ii) If $A, B \in M_{2\times2}(\mathbb{C})$ and $A \sim B$ then there exists $P \in M_{2\times2}(\mathbb{C})$ such that $A = P^{-1}BP$ which implies $PAP^{-1} = B$ and thus $B \sim A$.

(iii) Suppose $A, B, C \in M_{2\times2}(\mathbb{C})$ and $A \sim B$ and $B \sim C$. Therefore, there exists $P, Q \in M_{2\times2}(\mathbb{C})$ such that $A = P^{-1}BP$ and $B = Q^{-1}CQ$. Therefore, $A = P^{-1}Q^{-1}CQP = (QP)^{-1}CQP$ and thus $A \sim C$.

By items (i)-(iii), similarity between matrices is an equivalence relationship.

$\blacksquare$

#6

Let $A \in M_{2\times2}(\mathbb{C})$.

(a) Show that A is similar to I if and only if $A = I$.

(b) Show that A is similar to 0 if and only if $A = 0$.

Solution:

$A \sim I \Leftrightarrow$ there exists $P \in M_{2\times2}(\mathbb{C})$ such that $A = P^{-1}IP = I$.

$A \sim 0 \Leftrightarrow$ there exists $0 \in M_{2\times2}(\mathbb{C})$ such that $A = P^{-1}0P = 0$.

$\blacksquare$

#7

Let $A \in M_{n \times n}(\mathbb{C})$.

(a) Show that the linear system $A\vec{x}=\vec{y}$ has a unique solution for some $\vec{y} \in \mathbb{C}^n$ if and only if has a solution for all $y \in \mathbb{C}^n$.

(b) Show that the linear system $A\vec{x}=\vec{y}$ has a unique solution for all $\vec{y} \in \mathbb{C}^n$ if and only if the only solution to $A\vec{x}=0$ is $\vec{x}=0$.

Solution:

(a) Suppose $A\vec{x}=\vec{y}$ has a unique solution $\vec{x}^* \in \mathbb{C}^n$ for some $\vec{y}^* \in \mathbb{C}^n$. Now, suppose $\bar{x} \in \ker(A)$. Therefore,

$$A(\vec{x}^*+\bar{x}) = A\vec{x}^* + A\bar{x} = A\vec{x}^* = \vec{y}^*$$

which is a contradiction unless $\bar{x}=0$. Consequently,

$$\text{nullity}(A)=0 \Rightarrow \text{rank}(A)=n \Rightarrow \text{im}(A)=\mathbb{C}^n$$

and thus $A\vec{x}=\vec{y}$ has a solution for all $\vec{x}$.

(b) ($\Rightarrow$) Suppose $A\vec{x}=\vec{y}$ has a unique solution. Now, suppose there exists $\vec{x}^* \in \mathbb{C}^n$ and $\vec{y}^* \in \mathbb{C}^n$ such that $A\vec{x}^*=\vec{y}^*$. If $\bar{x} \in \ker(A)$ then $A(\vec{x}+\bar{x}) = A\vec{x}^* = \vec{y}^*$ which is a contradiction unless $\bar{x}=0$. Therefore, $\ker(A)=0$ which implies the only solution to $A\bar{x}=0$ is $\bar{x}=0$.

($\Leftarrow$) If the only solution to $A\vec{x}=0$ is $\vec{x}=0$ then $\ker(A)=0$. Consequently, $\text{nullity}(A)=0 \Rightarrow \text{rank}(A)=n \Rightarrow \text{im}(A)=\mathbb{C}^n$ and thus A is onto. Since $\ker(A)=0$, A is also one-to-one and thus for $\vec{y}$, there is a solution to $A\vec{x}=\vec{y}$ and it is unique.

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