MTH 225
Homework #6

Due Date: March 05, 2025

1. Suppose $A \in M_{n\times n}(\mathbb{R})$ has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 3$ with corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Find the matrix representation of the linear transformation $L[\mathbf{v}] = A\mathbf{u}$ with respect to following bases.
   (a) The standard basis.
   (b) The basis of eigenvectors.
   (c) The basis $\mathcal{A} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$.
2. Let $\mathbf{u} \in \mathbb{R}^n$ and $\mathbf{v} \in \mathbb{R}^n$ and $A = \mathbf{u}\mathbf{v}^T$.
   (a) Find the columns of A in terms of $\mathbf{u}$ and $\mathbf{v}$.
   (b) Show that $\text{rank}(A) = 1$.
3. Suppose $P \in M_{n\times n}(\mathbb{C})$ satisfies $P^2 = P$. Find all of the possible eigenvalues of P.
4. Suppose $N \in M_{n\times n}(\mathbb{C})$ satisfies $N^k = 0$ for some $k \in \mathbb{N}$. Find all possible eigenvalues of N.
5. Suppose $A \in M_{n\times n}(\mathbb{C})$ satisfies $A^4 = I$. Find all possible eigenvalues of A.
6. Suppose $A \in M_{3\times 3}(\mathbb{R})$ has $\lambda = 1$ as its only eigenvalue.
   (a) What is the algebraic multiplicity of λ?
   (b) Find an example of such a matrix A in which the geometric multiplicity of A is 1.
   (c) Find an example of such a matrix A in which the geometric multiplicity of A is 2.
   (d) Find an example of such a matrix A in which the geometric multiplicity of A is 3
7. Suppose $T : V \mapsto V$ is an invertible linear transformation and suppose $\lambda \neq 0$ is an eigenvalue with corresponding eigenvector $\mathbf{v}$.
   (a) Prove that $\lambda \neq 0$.
   (b) Prove that $\frac{1}{\lambda}$ is an eigenvalue of T^{-1} with corresponding eigenvector $\mathbf{v}$.
8. Suppose that $A \in M_{n\times n}(\mathbb{C})$ is diagonalizable.
   (a) Show that $cA + dI$, where c, d are any scalars, is diagonalizable.
   (b) Show that A^2 is diagonlizable.
   (c) Give an example of a matrix A that is not diagonalizable but A^2 is.

9. Suppose $A, B \in M_{n\times n}(\mathbb{C})$ are diagonlizable matrices with the same eigenspaces (but not necessarily the same eigenvalues). Prove that $AB = BA$.

10. Suppose $A \in M_{n\times n}(\mathbb{C})$ is a diagonlizable matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$.

    (a) Prove that $\det(A) = \lambda_1 \cdots \lambda_n$.

    (b) Prove that $\operatorname{tr}(A) = \lambda_1 + \ldots + \lambda_n$.

    (c) If $A \in M_{2\times 2}(\mathbb{C})$ is a diagonlizable, show that its eigenvalues λ_1, λ_2 satisfy

$$\lambda_{1,2} = \frac{1}{2}\left(\operatorname{tr}(A) \pm \sqrt{\operatorname{tr}(A)^2 - 4\det(A)}\right).$$

11. Suppose $T : P_4(\mathbb{R}) \mapsto P_4(\mathbb{R})$ is defined by $T(p(x)) = x\frac{dp}{dx}$.

    (a) Show that T is a linear transformation.

    (b) Find a basis for $\ker(T)$.

    (c) Find a basis for $\operatorname{im}(T)$.

    (d) Find all eigenvectors and eigenvalues of T. **Hint:** Don't try to do this using matrices.

    (e) Find the matrix representation of T with respect to the standard basis $S = \{1, x, x^2, x^3, x^4\}$.

12. Define the shift map $S : \mathbb{C}^n \mapsto \mathbb{C}^n$ by

$$S\left(\left[a_1, a_2, \ldots, a_{n-1}, a_n\right]^T\right) = \left[a_2, a_3, \ldots, a_n, a_1\right]^T.$$

    (a) Prove that S is a linear transformation.

    (b) Prove that the vectors $\omega_0, \omega_1, \ldots, \omega_n$ defined by

$$\omega_k = \left[1, e^{2k\pi i/n}, e^{4k\pi i/n}, \ldots, e^{2(n-1)k\pi i/n}\right]^T,$$

    form a basis of eigenvectors of S. What are the corresponding eigenvalues?

    (c) Find the matrix representation of S with respect to the standard basis.

13. Let $T : \mathbb{R}^\infty \mapsto \mathbb{R}^\infty$ and $S : \mathbb{R}^\infty \mapsto \mathbb{R}^\infty$ be defined by

$$T(a_0, a_1, a_2, \ldots) = (0, a_0, a_1, a_2, \ldots) \text{ and } S(a_0, a_1, a_2, \ldots) = (a_1, a_2, a_3, \ldots).$$

    (a) Show that T and S are linear transformations.

    (b) Show that $T \circ S$ is the identity transformation but $S \circ T$ is not.

    (c) Show that T has no eigenvalues.

    (d) Find all of the eigenvalues and eigenvectors of S.

Homework #6

#1

Suppose $A \in M_{n\times n}(\mathbb{R})$ has eigenvalues $\lambda_1 = -1$, $\lambda_2 = 3$ with corresponding eigenvectors $\vec{v}_1 = \begin{bmatrix}1\\2\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}2\\3\end{bmatrix}$. Find the matrix representation of the linear transformation $L[\vec{v}] = A\vec{v}$ with respect to the following bases.

(a) The standard basis.

(b) The basis of eigenvectors

(c) The basis $\mathcal{A} = \left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}\right\}$.

Solution:

(a) $$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = T\left(2\begin{bmatrix}2\\3\end{bmatrix} - 3\begin{bmatrix}1\\2\end{bmatrix}\right)$$
$$= 2T\left(\begin{bmatrix}2\\3\end{bmatrix}\right) - 3T\left(\begin{bmatrix}1\\2\end{bmatrix}\right)$$
$$= 6\cdot\begin{bmatrix}2\\3\end{bmatrix} + 3\begin{bmatrix}1\\2\end{bmatrix}$$
$$= \begin{bmatrix}15\\24\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = T\left(2\begin{bmatrix}1\\2\end{bmatrix} - \begin{bmatrix}2\\3\end{bmatrix}\right)$$
$$= 2T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) - T\left(\begin{bmatrix}2\\3\end{bmatrix}\right)$$
$$= -2\begin{bmatrix}1\\2\end{bmatrix} - 3\begin{bmatrix}2\\3\end{bmatrix}$$
$$= \begin{bmatrix}-8\\-13\end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix}15 & -8\\24 & -13\end{bmatrix}$$

(b) If $\beta = \left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\3\end{bmatrix}\right\}$ then
$$[T(\beta,\beta)] = \begin{bmatrix}-1 & 0\\0 & 3\end{bmatrix}.$$

(c) $$I[(\mathcal{A}, S)] = \begin{bmatrix}1 & 3\\1 & 4\end{bmatrix}$$

$$\Rightarrow I[(S,\mathcal{A})] = \begin{bmatrix}1 & 3\\1 & 4\end{bmatrix}^{-1} = \begin{bmatrix}4 & -3\\-1 & 1\end{bmatrix}$$

Therefore,

$$[T(A,A)] = I[(S,A)]\,T[(S,S)]\,I[(A,S)]$$

$$= \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 15 & -8 \\ 24 & -13 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 7 & 13 \\ 11 & 20 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & -8 \\ 4 & 7 \end{bmatrix}.$$

#2

Let $\vec{u} \in \mathbb{R}^n$ and $\vec{v} \in \mathbb{R}^n$ and $A = \vec{u}\vec{v}^T$.

(a) Find the columns of A in terms of $\vec{u}$ and $\vec{v}$.

(b) Show that $\text{rank}(A) = 1$.

Solution:

(a) Computing, we have

$$\begin{aligned} A\vec{e}_i &= \vec{u}\vec{v}^T\vec{e}_i \\ &= \vec{u}v_i \\ &= v_i\vec{u} \end{aligned}$$

Therefore, $A = [v_1\vec{u} | \cdots | v_n\vec{u}]$.

(b) $\text{rank}(A) = \dim(\text{im}(A)) = \dim(\text{col}(A)) = 1$.

$\blacksquare$

#3

Suppose $P \in M_{n\times n}(\mathbb{C})$ satisfies $P^2 = P$. Find all the possible eigenvalues of P.

Solution:

Let $\vec{v} \in \mathbb{C}^n$ be an eigenvector with corresponding eigenvalue λ. Therefore,

$$\lambda^2\vec{v} = P^2\vec{v} = P\vec{v} = \lambda\vec{v}$$

and thus $\lambda^2 = \lambda$. Consequently, $\lambda = 0, 1$.

$\blacksquare$

#4

Suppose $N \in M_{n\times n}(\mathbb{C})$ satisfies $N^k = 0$ for some $k \in \mathbb{N}$. Find all possible eigenvalues of N.

Solution:

Let $\lambda \in \mathbb{C}$ be an eigenvalue with corresponding eigenvector $\vec{v}$. Therefore, $\lambda^k\vec{v} = N^k\vec{v} = 0 \Rightarrow \lambda = 0$. $\blacksquare$

#5

Suppose $A \in M_{n\times n}(\mathbb{C})$ satisfies $A^4 = I$. Find all possible eigenvalues of A.

Solution:

Let $\vec{v}$ be an eigenvector with corresponding eigenvalue $\lambda \in \mathbb{C}$. Therefore,

$$\vec{v} = I\vec{v} = A^4\vec{v} = \lambda^4\vec{v}$$

$$\Rightarrow \lambda^4 = 1$$

$$\Rightarrow \lambda = \pm 1, \pm i.$$

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#6

Suppose $A \in M_{3\times 3}(\mathbb{R})$ has $\lambda = 1$ as its only eigenvalue.

(a) What is the algebraic multiplicity of λ?

Find examples of such an A in which the geometric multiplicity is

(b) 1

(c) 2

(d) 3

Solution:

(a) 3

(b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

(d) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

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#7

Suppose $T: V \to V$ is an invertible linear transformation and suppose $\lambda \neq 0$ is an eigenvalue with corresponding eigenvector $\vec{v}$.

(a) Prove that $\lambda \neq 0$.

(b) Prove that $1/\lambda$ is an eigenvalue of T^{-1} with corresponding eigenvector $\vec{v}$.

Solution:

(a) If $\lambda = 0$ then $T\vec{v} = 0 \cdot \vec{v} = 0$ which implies $\vec{v} \in \ker(A)$. However, since T is invertible it follows that $\ker(A) = \{0\}$ which is a contradiction. Therefore, $\lambda \neq 0$.

(b) Since $T\vec{v} = \lambda \vec{v}$ it follows that $\vec{v} = \lambda T^{-1}\vec{v}$ and thus

$$T^{-1}\vec{v} = 1/\lambda\, \vec{v}.$$

$\blacksquare$

#8

Suppose $A \in M_{n \times n}(\mathbb{C})$ is diagonalizable.

(a) Show that $cA + dI$ is diagonalizable.

(b) Show that A^2 is diagonalizable.

(c) Give an example of a matrix A that is not diagonalizable but A^2 is.

Solution:

If A is diagonalizable then there exists a diagonal matrix Λ and an invertible matrix X such that

$$A = X \Lambda X^{-1}.$$

(a) $cA + dI = cX\Lambda X^{-1} + dXX^{-1} = X(c\Lambda + dI)X^{-1}$

which proves that $cA + dI$ is diagonalizable.

(b) $A^2 = A \cdot A = X \Lambda X^{-1} X \Lambda X^{-1} = X \Lambda^2 X^{-1}$
which proves that A^2 is diagonalizable.
(c) Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Therefore, the eigenvalues are $\lambda = 0$ with eigenspace $E_0 = \operatorname{span}\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\}$. However, $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ which has eigenvalue $\lambda = 0$ with eigenspan $E_0 = \mathbb{R}^2$.

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#9

Suppose $A, B \in M_{n\times n}(\mathbb{C})$ are diagonlizable with the same eigenspaces. Prove that $AB = BA$.

Solution:

If A, B are diagonalizable with the same eigenspaces then there exists $X \in M_{n\times n}(\mathbb{C})$ and diagonal matrices $\Lambda_1, \Lambda_2 \in M_{n\times n}(\mathbb{C})$ such that

$$A = X \Lambda_1 X^{-1}, \quad B = X \Lambda_2 X^{-1}.$$

Therefore,

$$\begin{aligned} AB &= X \Lambda_1 X^{-1} X \Lambda_2 X^{-1} \\ &= X \Lambda_1 \Lambda_2 X^{-1} \\ &= X \Lambda_2 \Lambda_1 X^{-1} \\ &= X \Lambda_2 X^{-1} X \Lambda_1 X^{-1} \\ &= BA. \end{aligned}$$

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#10

Suppose $A \in M_{n\times n}(\mathbb{C})$ is diagonlizable with eigenvalues $\lambda_1, \dots, \lambda_n$.

(a) Prove that $\det(A) = \lambda_1 \cdots \lambda_n$

(b) Prove that $\mathrm{tr}(A) = \lambda_1 + \dots + \lambda_n$.

(c) If $A \in M_{2\times 2}(\mathbb{C})$ show that its eigenvalues λ_1, λ_2 satisfy

$$\lambda_{1,2} = \frac{1}{2}\left(\mathrm{tr}(A) \pm \sqrt{\mathrm{tr}(A)^2 - 4\det(A)}\right).$$

Solution:

Since A is diagonalizable there exists $X \in M_{n\times n}(\mathbb{C})$ and diagonal matrix Λ such that $A = X \Lambda X^{-1}$.

(a) $\det(A) = \det(\Lambda) = \lambda_1 \cdots \lambda_n$.

(b) $\mathrm{tr}(A) = \mathrm{tr}(\Lambda) = \lambda_1 + \dots + \lambda_n$.

(c) $\det(A) = \lambda_1 \lambda_2$ and $\lambda_1 + \lambda_2 = \mathrm{tr}(A)$. Therefore,

$$\lambda_1 = \mathrm{tr}(A) - \lambda_2$$

$$\Rightarrow \det(A) = \lambda_2(\mathrm{tr}(A) - \lambda_2)$$

$$\Rightarrow \lambda_2^2 - \mathrm{tr}(A)\lambda_2 + \det(A) = 0$$

$$\Rightarrow \lambda_2 = \frac{\mathrm{tr}(A) \pm \sqrt{\mathrm{tr}(A)^2 - 4\det(A)}}{2}$$

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#11

Suppose $T: P_4(\mathbb{R}) \to P_4(\mathbb{R})$ is defined by $T(p(x)) = x \cdot \frac{dp}{dx}$.

(a) Show that T is a linear transformation.

(b) Find a basis for $\ker(T)$.

(c) Find a basis for $\mathrm{im}(T)$.

(d) Find all eigenvectors and eigenvalues of T.

(e) Find the matrix representation of T with respect to the standard basis $\{1, x, x^2, x^3, x^4\}$.

Solution:

(a) If we let $p, q \in P_4(\mathbb{R})$ and $\lambda \in \mathbb{R}$ then

$$T(p+\lambda q) = x\frac{d}{dx}(p+\lambda q)$$
$$= x\frac{dp}{dx} + \lambda x \frac{dq}{dx}$$
$$= T(p) + \lambda T(q)$$

and thus T is linear.

(b) $\ker(T) = \operatorname{span}\{1\}$.

(c) $\operatorname{im}(T) = \operatorname{span}\{x, x^2, x^3, x^4\}$.

(d) The eigenvectors are $\{1, x, x^2, x^3, x^4\}$ with eigenvalues $0, 1, 2, 3, 4$ respectively.

(e) $[T(S,S)] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$.

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#12

Define the shift map $S: \mathbb{C}^n \to \mathbb{C}^n$ by

$$S([a_1, a_2, \ldots, a_{n-1}, a_n]^T) = [a_2, a_3, \ldots, a_n, a_1]^T.$$

(a) Prove that T is a linear transformation.

(b) Prove that the vectors $w_0, w_1, \ldots, w_n$ defined by

$$w_k = [1, e^{2k\pi i/n}, e^{4k\pi i/n}, \ldots, e^{2(n-1)k\pi i/n}]^T$$

form a basis of eigenvectors of S. What are the corresponding eigenvalues.

(c) Find the matrix representation of S.

Solution:

(a) Letting $\vec{v}=[v_1,\ldots,v_n]^T, \vec{u}=[u_1,\ldots,u_n]^T \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$.

Therefore,

$$S(\vec{v}+\lambda\vec{u})=S([v_1+\lambda u_1,\ldots,v_n+\lambda u_n]^T)$$
$$=[v_2+\lambda u_2,\ldots,v_1+\lambda u_1]^T$$
$$=[v_2,\ldots,v_1]^T+\lambda[u_2,\ldots,u_1]^T$$
$$=S(\vec{v})+\lambda S(\vec{u})$$

and thus S is linear.

(b) $S(w_k)=[e^{2\pi ik/n}, e^{4\pi ik/n},\ldots, e^{2(n-1)k\pi i/n}, 1]^T$

$$=e^{2\pi ik/n}[1, e^{2\pi ik/n},\ldots,e^{2(n-2)k\pi i/n}, e^{-2\pi ik/n}]^T$$
$$=e^{2\pi ik/n}[1, e^{2\pi ik/n},\ldots,e^{2(n-1)k\pi i/n}]^T$$
$$=e^{2\pi ik/n}w_k.$$

Therefore, w_k is an eigenvector with corresponding eigenvalue $e^{2\pi ik/n}$.

(c) $[S]=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$

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#13.

Let $T:\mathbb{R}^\infty \to \mathbb{R}^\infty$ and $S:\mathbb{R}^\infty \to \mathbb{R}^\infty$ be defined by

$$T(a_0,a_1,a_2,\ldots)=(0,a_0,a_1,a_2,\ldots) \text{ and } S(a_0,a_1,a_2,\ldots)=(a_1,a_2,a_3,\ldots).$$

(c) Show that T has no eigenvalues.

(d) Find all of the eigenvalues and eigenvectors of S.

Solution:

(a) Suppose $T(a_0,a_1,a_2,\ldots)=\lambda(a_0,a_1,a_2,\ldots)$. Therefore,

$$(0,a_0,a_1,a_2,\ldots)=\lambda(a_0,a_1,a_2,\ldots)$$

If $\lambda\neq 0$ then

$a_0=0$, $a_2=1/\lambda\, a_1=0,\ldots$

$a_1=1/\lambda\, a_0=0$

But $(0,0,0,\dots)$ cannot be an eigenvector. If instead $\lambda=0$ we have

$$(0,a_0,a_1,a_2,\dots)=(0,0,0,\dots)$$

which again only has $(0,0,0,\dots)$ as the solution. Thus there are no eigenvectors.

(b) Suppose $S(a_0,a_1,a_2,\dots)=\lambda(a_0,a_1,a_2,\dots)$. Therefore,

$$(a_1,a_2,a_3,\dots)=\lambda(a_0,a_1,a_2,\dots)$$

Consequently, for $\lambda\neq 0$ we have

$$\begin{aligned} a_1 &= \lambda a_0 \\ a_2 &= \lambda a_1 = \lambda^2 a_0 \\ a_3 &= \lambda a_2 = \lambda^3 a_0 \\ &\vdots \\ a_n &= \lambda^n a_0 \\ &\vdots \end{aligned}$$

Therefore, for all $\lambda\neq 0$, λ is an eigenvalue with corresponding eigenspace

$$E_\lambda=\operatorname{span}\{1,\lambda,\lambda^2,\dots\}.$$

If $\lambda=0$ we have

$$(a_1,a_2,a_3,\dots)=(0,0,0,\dots).$$

Therefore,

$$E_0=\operatorname{span}\{1,0,0,\dots\}.$$

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