MTH 225
Homework #7

Due Date: March 19, 2025

1. Prove that if $b > 1$ then the function $\langle \cdot, \cdot \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \mapsto \mathbb{R}$ defined by
$$\langle \mathbf{v}, \mathbf{w} \rangle = v_1 w_1 - v_1 w_2 - v_2 w_1 + b v_2 w_2$$
is an inner product on $\mathbb{R}^2$.

2. Show that the function $\langle \cdot, \cdot \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \mapsto \mathbb{R}$ defined by
$$\langle \mathbf{v}, \mathbf{w} \rangle = v_1 w_1 + v_1 w_2 + v_2 w_1 + v_2 w_2$$
does not define an inner product on $\mathbb{R}^2$.

3. Let V be a vector space with a complex inner product $\langle \cdot, \cdot \rangle : V \times V \mapsto \mathbb{C}$.
   (a) Prove that $\langle \mathbf{x}, \mathbf{v} \rangle = 0$ for all $\mathbf{v} \in V$ if and only if $\mathbf{x} = 0$.
   (b) Prove that $\langle \mathbf{x}, \mathbf{v} \rangle = \langle \mathbf{y}, \mathbf{v} \rangle$ for all $\mathbf{v} \in V$ if and only if $\mathbf{x} = \mathbf{y}$.
   (c) Let $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis for V. Prove that $\langle \mathbf{x}, \mathbf{v}_i \rangle = \langle \mathbf{y}, \mathbf{v}_i \rangle$, $i = 1, \dots, n$, if and only if $\mathbf{x} = \mathbf{y}$.
   (d) Prove that $\mathbf{x} \in \mathbb{R}^n$ solves the linear system $A\mathbf{x} = \mathbf{b}$ if and only if
   $$\mathbf{x}^T A^T \mathbf{v} = \mathbf{b}^\mathbf{T} \mathbf{v}$$
   for all $\mathbf{v}$.

4. Let $V = M_{n \times n}(\mathbb{R})$. Prove that
$$\langle A, B \rangle = \operatorname{tr}(A^T B)$$
defines an inner product on V.

5. Let $V = C^1([0,1])$. Let $\langle \cdot, \cdot \rangle : V \times V \mapsto \mathbb{R}$ be defined by
$$\langle f, g \rangle = \int_0^1 (f(x)g(x) + f'(x)g'(x))\, dx.$$
   (a) Prove that $\langle \cdot, \cdot \rangle$ defines an inner product on V.
   (b) Write out an expression for the norm corresponding to this inner product.

6. Let $V = C^1([0,1])$ and $W = \{f \in C^1([0,1]) : f(0) = 0\}$. Let $\langle \cdot, \cdot \rangle : V \times V \mapsto \mathbb{R}$ be defined by

$$\langle f, g \rangle = \int_0^1 f'(x)g'(x)dx.$$

(a) Prove that $\langle \cdot, \cdot \rangle$ is not an inner product on V.

(b) Prove that $\langle \cdot, \cdot \rangle$ is an inner product on W.

7. Let V be a complex inner product space with inner product $\langle \cdot, \cdot \rangle : V \times V \mapsto \mathbb{C}$.

(a) If $z \in \mathbb{C}$, prove that $z + \overline{z} = 2\text{Re}(z)$ and $z - \overline{z} = 2i\text{Im}(z)$.

(b) Prove for all $\mathbf{v}, \mathbf{w} \in V$ that

$$\|\mathbf{v} + \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + 2\text{Re}(\langle \mathbf{v}, \mathbf{w} \rangle) + \|\mathbf{w}\|^2.$$

(c) Prove for all $\mathbf{v}, \mathbf{w} \in V$ that

$$\|\mathbf{v} - i\mathbf{w}\|^2 = \|\mathbf{v}\|^2 + 2\text{Im}(\langle \mathbf{v}, \mathbf{w} \rangle) - \|\mathbf{w}\|^2.$$

(d) Prove for all $\mathbf{v}, \mathbf{w} \in V$ that

$$\langle \mathbf{v}, \mathbf{w} \rangle = \frac{1}{4}\left(\|\mathbf{v} + \mathbf{w}\|^2 - \|\mathbf{v} - \mathbf{w}\|^2 - i\|\mathbf{v} + i\mathbf{w}\|^2 + i\|\mathbf{v} - i\mathbf{w}\|^2\right).$$

8. Let $\mathbf{v}, \mathbf{w} \in V$ and suppose $\langle \cdot, \cdot \rangle$ is a real inner product on V. Prove that

$$\|\mathbf{v} + \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2$$

if and only if $\mathbf{v}$ and $\mathbf{w}$ are orthogonal. How does this result change if $\langle \cdot, \cdot \rangle$ is a complex inner product?

9. Let $\mathbf{v}, \mathbf{w} \in V$ be a basis for V and suppose that $\langle \cdot, \cdot \rangle$ is an inner product on V. Suppose further that $\|\mathbf{v}\| = 1$ and $\|\mathbf{w}\| = 1$.

(a) If $\langle \cdot, \cdot \rangle$ is a real-valued inner product, prove that $\mathbf{v} + \mathbf{w}$ and $\mathbf{v} - \mathbf{w}$ forms an orthogonal basis for V.

(b) How does this result change if $\langle \cdot, \cdot \rangle$ is a complex-valued inner product?

10. Prove that the polynomials $p_0(x) = 1$, $p_1(x) = x$, $p_2(x) = x^2 - \frac{1}{3}$, and $p_3(x) = x^3 - \frac{3}{5}x$ form an orthogonal basis for $P_2(\mathbb{R})$ with respect to the following inner product:

$$\langle f, g \rangle = \int_{-1}^1 f(x)g(x)dx.$$

11. Show for $k \in \mathbb{N}$ that the complex exponentials e^{ikx} are orthogonal under the inner product defined by

$$\langle f, g \rangle = \int_{-\pi}^{\pi} \overline{f(x)}g(x)dx.$$

Homework #7

#1

Prove that if $b>1$ then the function $\langle \cdot , \cdot \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\langle \vec{v}, \vec{w} \rangle = v_1 w_1 - v_1 w_2 - v_2 w_1 + b v_2 w_2$$

is an inner product on $\mathbb{R}^2$.

Solution:

1. Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^2$. Therefore,

$$\begin{aligned} \langle \vec{u}+\vec{v}, \vec{w} \rangle &= (u_1+v_1)w_1 - (u_1+v_1)w_2 - (u_2+v_2)w_1 + b(u_2+v_2)w_2 \\ &= u_1w_1 - u_1w_2 - u_2w_1 + bu_2w_2 + v_1w_1 - v_1w_2 - v_2w_1 + bv_2w_2 \\ &= \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle \end{aligned}$$

2. Let $\vec{u}, \vec{v} \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$. Therefore,

$$\begin{aligned} \langle \lambda \vec{u}, \vec{v} \rangle &= \lambda u_1 v_1 - \lambda u_1 v_2 - \lambda u_2 v_1 + \lambda b u_2 v_2 \\ &= \lambda (u_1v_1 - u_1v_2 - u_2v_1 + bu_2v_2) \\ &= \lambda \langle \vec{u}, \vec{v} \rangle. \end{aligned}$$

3. Let $\vec{u}, \vec{v} \in \mathbb{R}^2$. Therefore,

$$\begin{aligned} \langle \vec{u}, \vec{v} \rangle &= u_1v_1 - u_1v_2 - u_2v_1 + bu_2v_2 \\ &= v_1u_1 - v_1u_2 - v_2u_1 + bv_2u_2 \\ &= \langle \vec{v}, \vec{u} \rangle. \end{aligned}$$

4. Let $\vec{u} \in \mathbb{R}^2$. Therefore,

$$\begin{aligned} \langle \vec{u}, \vec{u} \rangle &= u_1^2 - 2u_1^2 + bu_1^2 \\ &= (b-1)u_1^2 \end{aligned}$$

and thus since $b>1$, $\langle \vec{u}, \vec{u} \rangle \geq 0$ and only equals 0 when $u_1 = 0$.

By items 1-4, $\langle \cdot , \cdot \rangle$ is an inner product on $\mathbb{R}^2$.

$\blacksquare$

#2

Show that the function $\langle\cdot,\cdot\rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\langle \vec{v}, \vec{w} \rangle = v_1 w_1 + v_1 w_2 + v_2 w_1 + v_2 w_2$$

does not define an inner product on $\mathbb{R}^2$.

Solution:

If we let $\vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ then

$$\langle \vec{v}, \vec{v} \rangle = 1 - 1 - 1 + 1 = 0$$

and thus since $\vec{v} \neq 0$ this cannot be an inner product.

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#3

Let V be a vector space with complex inner product $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{C}$

(a) Prove that $\langle \vec{x}, \vec{v} \rangle = 0$ for all $\vec{v} \in V$ if and only if $\vec{x} = 0$.

(b) Prove that $\langle \vec{x}, \vec{v} \rangle = \langle \vec{y}, \vec{v} \rangle$ for all $\vec{v} \in V$ if and only if $\vec{x} = \vec{y}$.

(c) Let $\{\vec{v}_1, \dots, \vec{v}_n\}$ be a basis for V. Prove that $\langle \vec{x}, \vec{v}_i \rangle = \langle \vec{y}, \vec{v}_i \rangle$ if and only if $\vec{x} = \vec{y}$.

(d) Prove that $\vec{x} \in \mathbb{R}^n$ solves the linear system $A\vec{x} = \vec{b}$ if and only if

$$\vec{x}^T A^T \vec{v} = \vec{b}^T \vec{v}$$

for all $\vec{v} \in V$.

Solution:

(a) If $\langle \vec{x}, \vec{v} \rangle = 0$ for all $\vec{v} \in V$ then $\langle \vec{x}, \vec{x} \rangle = 0$ which implies $\vec{x} = 0$.

If $\vec{x} = 0$ then $\langle \vec{x}, \vec{v} \rangle = \langle 0, \vec{v} \rangle = 0 \cdot \langle \vec{v}, \vec{v} \rangle = 0$.

(b) $\langle \vec{x}, \vec{v} \rangle = \langle \vec{y}, \vec{v} \rangle \Leftrightarrow \langle \vec{x} - \vec{y}, \vec{v} \rangle = 0 \Leftrightarrow \vec{x} - \vec{y} = 0 \Leftrightarrow \vec{x} = \vec{y}$.

(c) $\vec{x} = \vec{y} \Leftrightarrow \vec{x} = a_1 \vec{v}_1 + \dots + a_n \vec{v}_n = \vec{y} \Leftrightarrow \langle \vec{x}, \vec{v}_i \rangle = a_1 \langle \vec{v}_1, \vec{v}_i \rangle + \dots + a_n \langle \vec{v}_n, \vec{v}_i \rangle = \langle \vec{y}, \vec{v}_i \rangle$

(d) $\vec{x}^T A^T \vec{v} = \vec{b}^T \vec{v} \Leftrightarrow \langle A\vec{x}, \vec{v} \rangle = \langle b, \vec{v} \rangle \Leftrightarrow A\vec{x} = \vec{b}$.

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#4.

Let $V = M_{n\times n}(\mathbb{R})$. Prove that

$$\langle A,B\rangle = tr(A^T B)$$

defines an inner product on V.

Solution:

If we let a_{ij}, b_{ij} denote the components of A and B respectively then

$$\langle A,B\rangle = tr(A^T B) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ji}b_{ji}$$

1. Let $A, B, C \in M_{n\times n}(\mathbb{R})$. Therefore,

$$\langle A+B, C\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n}(a_{ji}+b_{ji})c_{ji} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ji}c_{ji} + \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ji}c_{ji}$$

$$\Rightarrow \langle A+B, C\rangle = \langle A, C\rangle + \langle B, C\rangle.$$

2. Let $\lambda \in \mathbb{R}$. Therefore,

$$\langle \lambda A, B\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda a_{ji}b_{ji} = \lambda \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ji}b_{ji} = \lambda\langle A, B\rangle.$$

3. Let $A, B \in M_{n\times n}(\mathbb{R})$. Therefore,

$$\langle A, B\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ji}b_{ji} = \sum_{j=1}^{n}\sum_{i=1}^{n} b_{ji}a_{ji} = \langle B, A\rangle.$$

4. $\langle A, A\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ji}a_{ji} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ji}^2 \geq 0$ and equals zero if and only if $a_{ij}=0$ for all i,j.

By items 1-4, $\langle \cdot, \cdot \rangle$ is an inner product.

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#6

Let $V=C^1([0,1])$ and $W=\{f\in C^1([0,1]): f(0)=0\}$. Let $\langle\cdot,\cdot\rangle: V\times V\to\mathbb{R}$ be defined by

$$\langle f,g\rangle=\int_0^1 f'(x)g'(x)dx.$$

(a) Prove that $\langle\cdot,\cdot\rangle$ is not an inner product on V.

(b) Prove that $\langle\cdot,\cdot\rangle$ is an inner product on W.

Solution:

(a) If $f=1$ then $\langle f,f\rangle=\int_0^1 0\,dx=0$. Since $f\neq 0$ it follows that $\langle\cdot,\cdot\rangle$ is not an inner product on V.

(b) Properties 1-3 are straightforward. Now if $f\in W$ then

$$\langle f,f\rangle=\int_0^1 f'(x)^2dx\geq 0$$

and is only 0 if $f'(x)=0$ which implies f is a constant. Since $f(0)=0$ this implies $f=0$.

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#7.

Let V be a complex inner product space with inner product $\langle\cdot,\cdot\rangle: V\times V\to\mathbb{C}$

(a) If $z\in\mathbb{C}$, prove that $z+\bar{z}=2\mathrm{Re}(z)$ and $z-\bar{z}=2i\,\mathrm{Im}(z)$.

(b) Prove for all $\vec{v},\vec{w}\in V$ that

$$\|\vec{v}+\vec{w}\|^2=\|\vec{v}\|^2+2\mathrm{Re}(\langle\vec{v},\vec{w}\rangle)+\|\vec{w}\|^2$$

(c) Prove for all $\vec{v},\vec{w}\in V$ that

$$\|\vec{v}-i\vec{w}\|^2=\|\vec{v}\|^2+2\,\mathrm{Im}(\langle\vec{v},\vec{w}\rangle)-\|\vec{w}\|^2$$

(d) Prove for all $\vec{v},\vec{w}\in V$ that

$$\langle\vec{v},\vec{w}\rangle=\tfrac{1}{4}\left(\|\vec{v}+\vec{w}\|^2-\|\vec{v}-\vec{w}\|^2-i\|\vec{v}+i\vec{w}\|^2+i\|\vec{v}-i\vec{w}\|^2\right)$$

Solution:

(a) Let $z = Re(z) + i\,Im(z)$. Therefore,

$$z + \bar{z} = Re(z) + i\,Im(z) + Re(z) - i\,Im(z) = 2Re(z)$$

$$z - \bar{z} = Re(z) + i\,Im(z) - Re(z) + i\,Im(z) = 2i\,Im(z)$$

(b) $$\|\vec{v}+\vec{w}\|^2 = \langle \vec{v}+\vec{w}, \vec{v}+\vec{w}\rangle$$
$$= \langle \vec{v},\vec{v}\rangle + \langle \vec{w},\vec{v}\rangle + \overline{\langle \vec{v},\vec{w}\rangle} + \langle \vec{w},\vec{w}\rangle$$
$$= \|\vec{v}\|^2 + \langle \vec{v},\vec{w}\rangle + \overline{\langle \vec{v},\vec{w}\rangle} + \|\vec{w}\|^2$$
$$= \|\vec{v}\|^2 + 2Re(\langle \vec{v},\vec{w}\rangle) + \|\vec{w}\|^2$$

(c) $$\|\vec{v}-i\vec{w}\|^2 = \langle \vec{v}-i\vec{w}, \vec{v}-i\vec{w}\rangle$$
$$= \langle \vec{v},\vec{v}\rangle + \langle \vec{v},-i\vec{w}\rangle + \langle -i\vec{w},\vec{v}\rangle + \langle -i\vec{w},-i\vec{w}\rangle$$
$$= \|\vec{v}\|^2 - i\langle \vec{v},\vec{w}\rangle + i\langle \vec{w},\vec{v}\rangle + i\cdot(-i)\langle \vec{w},\vec{w}\rangle$$
$$= \|\vec{v}\|^2 - i\langle \vec{v},\vec{w}\rangle + i\,\overline{\langle \vec{v},\vec{w}\rangle} + \|\vec{w}\|^2$$
$$= \|\vec{v}\|^2 - i(\langle \vec{v},\vec{w}\rangle - \overline{\langle \vec{v},\vec{w}\rangle}) + \|\vec{w}\|^2$$
$$= \|\vec{v}\|^2 - i\,2i\,Im(\langle \vec{v},\vec{w}\rangle) + \|\vec{w}\|^2$$
$$= \|\vec{v}\|^2 + 2\,Im(\langle \vec{v},\vec{w}\rangle) + \|\vec{w}\|^2$$

(d) $$\|\vec{v}+\vec{w}\|^2 - \|\vec{v}-\vec{w}\|^2 - i\|\vec{v}+i\vec{w}\|^2 + i\|\vec{v}-i\vec{w}\|^2$$
$$= \|\vec{v}\|^2 + 2Re(\langle \vec{v},\vec{w}\rangle) + \|\vec{w}\|^2 - \|\vec{v}\|^2 - 2Re(\langle \vec{v},-\vec{w}\rangle) - \|\vec{w}\|^2$$
$$- i(\|\vec{v}\|^2 + 2\,Im(\langle \vec{v},-\vec{w}\rangle) + \|\vec{w}\|^2) + i(\|\vec{v}\|^2 + 2\,Im(\langle \vec{v},\vec{w}\rangle) + \|\vec{w}\|^2)$$
$$= 4Re(\langle \vec{v},\vec{w}\rangle) + 4i\,Im(\langle \vec{v},\vec{w}\rangle)$$
$$= 4\langle \vec{v},\vec{w}\rangle$$
$$\Rightarrow \langle \vec{v},\vec{w}\rangle = \tfrac{1}{4}(\|\vec{v}+\vec{w}\|^2 - \|\vec{v}-\vec{w}\|^2 - i\|\vec{v}+i\vec{w}\|^2 + i\|\vec{v}-i\vec{w}\|^2)$$

$\blacksquare$

#8.

Let $\vec{v}, \vec{w} \in V$ and suppose $\langle\cdot,\cdot\rangle$ is a real inner product on V. Prove that $\|\vec{v}+\vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2$ if and only if $\vec{v}, \vec{w} \in V$ are orthogonal.

proof:

$$\|\vec{v}+\vec{w}\|^2 = \langle \vec{v}+\vec{w}, \vec{v}+\vec{w}\rangle$$
$$= \langle \vec{v},\vec{v}\rangle + 2\langle \vec{v},\vec{w}\rangle + \langle \vec{w},\vec{w}\rangle$$
$$= \|\vec{v}\|^2 + 2\langle \vec{v},\vec{w}\rangle + \|\vec{w}\|^2.$$

Consequently, $\|\vec{v}+\vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2$ if and only if $\langle \vec{v},\vec{w}\rangle = 0$.

#9

Let $\vec{v},\vec{w}\in V$ be a basis for V and suppose $\langle\cdot,\cdot\rangle$ is an inner product on V. Suppose further that $\|\vec{v}\|=1$ and $\|\vec{w}\|=1$.

(a) If $\langle\cdot,\cdot\rangle$ is a real valued inner product, prove that $\vec{v}+\vec{w}$ and $\vec{v}-\vec{w}$ forms an orthogonal basis for V.

(b) How does this result change if $\langle\cdot,\cdot\rangle$ is complex valued?

Solution:

(a) $$\langle \vec{v}+\vec{w}, \vec{v}-\vec{w}\rangle = \langle \vec{v},\vec{v}\rangle + \langle \vec{w},\vec{v}\rangle - \langle \vec{v},\vec{w}\rangle - \langle \vec{w},\vec{w}\rangle$$
$$= \|\vec{v}\|^2 + \langle \vec{v},\vec{w}\rangle - \langle \vec{v},\vec{w}\rangle - \|\vec{w}\|^2$$
$$= 1 + 0 - 1$$
$$= 0.$$

(b) If $\langle\cdot,\cdot\rangle$ is complex valued we have that
$$\langle \vec{v}+\vec{w}, \vec{v}-\vec{w}\rangle = \|\vec{v}\|^2 + \overline{\langle \vec{v},\vec{w}\rangle} - \langle \vec{v},\vec{w}\rangle - \|\vec{w}\|^2$$
$$= -2i\,\mathrm{Im}(\langle \vec{v},\vec{w}\rangle).$$

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#10

Prove that the polynomials $p_0(x)=1$, $p_1(x)=x$, $p_2(x)=x^2-1/3$, $p_3(x)=x^3-3/5x$ form an orthogonal basis for $P_2(\mathbb{R})$ with respect to the inner product

$$\langle f,g\rangle=\int_{-1}^{1} f(x)g(x)dx.$$

Solution:

Note, p_0, p_2 are even functions and p_1, p_3 are odd functions.

Therefore,

$$\langle p_0,p_1\rangle=\langle p_0,p_3\rangle=0$$

$$\langle p_2,p_1\rangle=\langle p_2,p_3\rangle=0$$

Therefore, we only need to compute $\langle p_0,p_2\rangle$ and $\langle p_1,p_3\rangle$.

$$\begin{aligned}\langle p_0,p_2\rangle&=\int_{-1}^{1}(x^2-1/3)dx\\&=2\int_{0}^{1}(x^2-1/3)dx\\&=2(1/3-1/3)\\&=0\end{aligned}$$

$$\begin{aligned}\langle p_1,p_3\rangle&=\int_{-1}^{1}(x^4-3/5x^2)dx\\&=2\int_{0}^{1}(x^4-3/5x^2)dx\\&=2(1/5-1/5)\\&=0\end{aligned}$$

$\blacksquare$

#11

Show for $k\in\mathbb{N}$ that the complex exponentials e^{ikx} are orthogonal under the inner product

$$\langle f,g\rangle=\int_{-\pi}^{\pi}\overline{f(x)}g(x)dx$$

Solution:

$$\langle e^{ikx},e^{imx}\rangle=\int_{-\pi}^{\pi}e^{-ikx}e^{imx}dx=\int_{-\pi}^{\pi}e^{i(m-k)x}dx=\frac{1}{i(m-k)}\left(e^{i(m-k)\pi}+e^{-i(m-k)\pi}\right)$$

$$\Rightarrow \langle e^{ikx}, e^{imx} \rangle = \frac{2}{m-k} \sin((m-k)\pi) = 0.$$

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