# MTH 225
# Homework #8

Due Date: March 26, 2025

1. Consider the following vectors

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix}, \mathbf{u}_2 = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\1\end{bmatrix}, \mathbf{u}_3 = \frac{1}{\sqrt{6}}\begin{bmatrix}-1\\1\\2\end{bmatrix}.$$

   (a) Show that $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ forms an orthonormal basis for $\mathbb{R}^3$ with respect to the standard inner product.

   (b) Find the coordinates of $\mathbf{v} = \begin{bmatrix}2 & 1 & 3\end{bmatrix}^T$ with respect to the above basis. **Hint:** Do not solve a $3 \times 3$ system of equations.

2. Let $\mathbf{u}_1, \ldots, \mathbf{u}_n$ be an orthonormal system in $\mathbb{C}^n$ with respect to the standard complex inner product. If $\mathbf{v} \in \mathbb{C}^n$, prove that

$$\|\mathbf{v}\|^2 = \sum_{i=1}^{n} |\langle \mathbf{v}, \mathbf{u}_i \rangle|^2,$$

   where $|\cdot|$ denotes the complex modulus.

3. Let $A \in M_{n\times n}(\mathbb{C})$. In these problems we assume the standard complex inner product on $\mathbb{C}^n$.

   (a) Prove for all $\mathbf{v}, \mathbf{w} \in \mathbb{C}^n$ that $\langle A\mathbf{v}, \mathbf{w}\rangle = \langle \mathbf{v}, A^*\mathbf{w}\rangle$.

   (b) Prove that $\ker(A^*) = (\mathrm{im}(A))^\perp$.

   (c) Prove that $\mathrm{im}(A^*) = (\ker(A))^\perp$

   (d) Prove that $\ker(A) = (\mathrm{im}(A^*))^\perp$

   (e) Prove that $\mathrm{im}(A) = (\ker(A^*))^\perp$.

   (f) Let $\mathbf{b} \in \mathbb{C}^n$. Prove there exists $\mathbf{v} \in \mathbb{C}^n$ that satisfies $A\mathbf{v} = \mathbf{b}$ if and only if $\langle \mathbf{b}, \mathbf{w}\rangle = 0$ for all $\mathbf{w}$ satisfying $A^*\mathbf{w} = 0$.

4. Let $A \in M_{n\times n}(\mathbb{C})$. In these problems we assume the standard complex inner product on $\mathbb{C}^n$.

   (a) Prove that $\ker(A) \subseteq \ker(A^*A)$

   (b) Prove that $\ker(A^*A) \subset \ker(A)$. **Hint:** If $\mathbf{v} \in \ker(A^*A)$, what is $\langle A\mathbf{v}, A\mathbf{v}\rangle$?

   (c) Prove that $\mathrm{rank}(A^*A) = \mathrm{rank}(A)$.

   (d) Prove that the columns of $A$ are linearly independent if and only if $A^*A$ is invertible.

5. Find an orthonormal basis, with respect to the standard inner product, for the following subspaces of $\mathbb{R}^4$:

(a) The span of the vectors

$$\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 2 \\ 1 \end{bmatrix}.$$

(b) The kernel of the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 & -1 \\ 3 & 2 & -1 & -1 \end{bmatrix}.$$

(c) The image of the matrix

$$B = \begin{bmatrix} 1 & -2 & 2 \\ 2 & -4 & 1 \\ 0 & 0 & -1 \\ -2 & 4 & 5 \end{bmatrix}.$$

(d) The set of all vectors orthogonal to the vector

$$\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}.$$

6. Let $V = P_\infty(\mathbb{R})$ be the vector space of polynomials of arbitrary large degree and $S \subset V$ be defined by $S = \text{span}\{1, t, t^2\}$. On $V$ define the inner product $\langle \cdot, \cdot \rangle$ by

$$\langle f, g \rangle = \int_0^\infty f(t)g(t)e^{-t}dt.$$

(a) Use integration by parts to show that for all $k \in \mathbb{N}$ that

$$\int_0^\infty t^k e^{-t} dt = k!.$$

(b) With respect to this inner product, prove that for $m, m \in \mathbb{N}$ that $\langle t^m, t^n \rangle = (m+n)!$ and $\|t^m\|^2 = (2m)!$.

(c) With respect to this inner product, find a set of orthonormal functions $\{q_0, q_1, q_2\}$ such that $\text{span}\{q_0, q_1, q_2\} = S$.

7. Show $A \in M_{2\times 2}(\mathbb{C})$ is unitary with respect to the standard complex inner product if and only if

$$A = \begin{bmatrix} a & b \\ -\bar{b}e^{i\phi} & \bar{a}e^{i\phi} \end{bmatrix},$$

for some $a, b \in \mathbb{C}$ satisfying $|a|^2 + |b|^2 = 1$ and $\phi \in \mathbb{R}$.

8. Suppose $A \in M_{n\times n}(\mathbb{C})$ is a unitary matrix with respect to the standard complex inner product. Prove that $A\mathbf{v} = \mathbf{b}$ has a solution if and only if $\mathbf{b} \in (\ker(A))^\perp$.

## Homework #8

### #1

Consider the following vectors

$$\vec{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix},\ \vec{v}_2 = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\1\end{bmatrix},\ \vec{v}_3 = \frac{1}{\sqrt{6}}\begin{bmatrix}-1\\1\\2\end{bmatrix}$$

(a) Show that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ forms an orthonormal basis for $\mathbb{R}^3$ with respect to the standard inner product.

(b) Find the coordinates of $\vec{v} = \begin{bmatrix}2\\1\\3\end{bmatrix}$ with respect to the above basis.

**Solution:**

(a) $\langle \vec{v}_1, \vec{v}_1 \rangle = \frac{1}{2}\cdot 2 = 1$

$\langle \vec{v}_1, \vec{v}_2 \rangle = \frac{1}{\sqrt{6}}\cdot(1-1) = 0$

$\langle \vec{v}_1, \vec{v}_3 \rangle = \frac{1}{\sqrt{12}}(-1+1) = 0$

$\langle \vec{v}_2, \vec{v}_2 \rangle = \frac{1}{3}(1+1+) = 1$

$\langle \vec{v}_2, \vec{v}_3 \rangle = \frac{1}{\sqrt{18}}(-1-1+2) = 0$

$\langle \vec{v}_3, \vec{v}_3 \rangle = \frac{1}{6}(1+1+4) = 1.$

Therefore, $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are orthonormal.

(b) $\langle \vec{v}, \vec{v}_1 \rangle = \frac{1}{\sqrt{2}}\cdot(2+1) = \frac{3}{\sqrt{2}}$

$\langle \vec{v}, \vec{v}_2 \rangle = \frac{1}{\sqrt{3}}(2-1+3) = \frac{4}{\sqrt{3}}$

$\langle \vec{v}, \vec{v}_3 \rangle = \frac{1}{\sqrt{6}}(-2+1+6) = \frac{5}{\sqrt{6}}$

Therefore, if we let $\beta = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ it follows that

$$[\vec{v}]_\beta = \begin{bmatrix} 3/\sqrt{2} \\ 4/\sqrt{3} \\ 5/\sqrt{6} \end{bmatrix}$$

#2

Let $\{\vec{U}_1, \ldots, \vec{U}_n\}$ be an orthonormal system in $\mathbb{C}^n$ with respect to the standard inner product. If $\vec{V} \in \mathbb{C}^n$, prove that

$$\|\vec{V}\|^2 = \sum_{i=1}^{n} |\langle \vec{V}, \vec{U}_i \rangle|^2.$$

proof:

If we let $\vec{V} = c_1\vec{U}_1 + \cdots + c_n\vec{U}_n$ then

$$\langle \vec{V}, \vec{U}_i \rangle = \langle c_1\vec{U}_1 + \cdots + c_n\vec{U}_n, \vec{U}_i \rangle$$
$$= \overline{c_i}$$

Therefore,

$$\vec{V} = \langle \vec{U}_1, \vec{V} \rangle \vec{U}_1 + \cdots + \langle \vec{U}_n, \vec{V} \rangle \vec{U}_n$$

and thus

$$\|\vec{V}\|^2 = \langle \vec{V}, \vec{V} \rangle$$
$$= \left\langle \sum_{i=1}^{n} \langle \vec{U}_i, \vec{V} \rangle \vec{U}_i, \sum_{j=1}^{n} \langle \vec{U}_j, \vec{V} \rangle \vec{U}_j \right\rangle$$
$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \langle \langle \vec{U}_i, \vec{V} \rangle \vec{U}_i, \langle \vec{U}_j, \vec{V} \rangle \vec{U}_j \rangle$$
$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \overline{\langle \vec{U}_i, \vec{V} \rangle} \langle \vec{U}_j, \vec{V} \rangle \langle \vec{U}_i, \vec{U}_j \rangle$$
$$= \sum_{i=1}^{n} \overline{\langle \vec{U}_i, \vec{V} \rangle} \langle \vec{U}_i, \vec{V} \rangle$$
$$= \sum_{i=1}^{n} |\langle \vec{U}_i, \vec{V} \rangle|^2$$
$$= \sum_{i=1}^{n} |\langle \vec{V}, \vec{U}_i \rangle|^2$$

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#3.

Let $A \in M_{n \times n}(\mathbb{C})$.

(a) Prove for all $\vec{v}, \vec{w} \in \mathbb{C}^n$ that $\langle A\vec{v}, \vec{w}\rangle = \langle \vec{v}, A^*\vec{w}\rangle$

(b) Prove that $\ker(A^*) = \operatorname{im}(A)^\perp$

(c) Prove that $\operatorname{im}(A^*) = \ker(A)^\perp$

(d) Prove that $\ker(A) = \operatorname{im}(A^*)^\perp$

(e) Prove that $\operatorname{im}(A) = \ker(A^*)^\perp$

(f) Let $\vec{b} \in \mathbb{C}^n$. Prove there exists $\vec{v} \in \mathbb{C}^n$ that satisfies $A\vec{v} = \vec{b}$ if and only if $\langle \vec{b}, \vec{w}\rangle = 0$ for all $\vec{w}$ satisfying $A^*\vec{w} = 0$.

Solution:

(a) $\langle A\vec{v}, \vec{w}\rangle = (A\vec{v})^*\vec{w} = \vec{v}^*A^*\vec{w} = \langle \vec{v}, A^*\vec{w}\rangle$

(b) ($\subseteq$) Suppose $\vec{v} \in \ker(A^*)$. Therefore, $A^*\vec{v} = 0$. Now if $\vec{y} \in \operatorname{im}(A)$ it follows that there exists $\vec{b}$ such that $\vec{y} = A\vec{b}$. Consequently,

$$\langle \vec{v}, \vec{y}\rangle = \langle \vec{v}, A\vec{b}\rangle = \langle A^*\vec{v}, \vec{b}\rangle = 0$$

and thus $\vec{v} \in \operatorname{im}(A)^\perp$.

($\supseteq$) Suppose $\vec{v} \in \operatorname{im}(A)^\perp$. Consequently, for all $\vec{w} \in \operatorname{im}(A)$ $\langle \vec{v}, \vec{w}\rangle = 0$. Therefore, for all $\vec{y} \in \mathbb{C}^n$

$$0 = \langle \vec{v}, A\vec{y}\rangle = \langle A^*\vec{v}, \vec{y}\rangle.$$

Since this is true for all $\vec{y} \in \mathbb{C}^n$ it follows that $A^*\vec{v} = 0$ which implies $\vec{v} \in \ker(A^*)$.

(c) By (b), $\ker((A^*)^*) = \operatorname{im}(A^*)^\perp \Rightarrow \ker(A)^\perp = \operatorname{im}(A^*)$.

(d) By (c), $\ker(A)^\perp = \operatorname{im}(A^*) \Rightarrow \ker(A) = \operatorname{im}(A^*)^\perp$

(e) By (b), $\ker(A^*) = \operatorname{im}(A)^\perp \Rightarrow \ker(A^*)^\perp = \operatorname{im}(A)$.

(f) $\vec{v} \in \mathbb{C}^n$ satisfies $A\vec{v} = \vec{b} \Leftrightarrow \vec{b} \in \operatorname{im}(A) \Leftrightarrow \vec{b} \in \ker(A^*)^\perp \Leftrightarrow \langle \vec{b}, \vec{w}\rangle = 0$ for all $\vec{w} \in \ker(A^*)$.

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#4

Let $A \in M_{n\times n}(\mathbb{C})$.

(a) Prove that $\ker(A) \subseteq \ker(A^*A)$.

(b) Prove that $\ker(A^*A) \subseteq \ker(A^*)$.

(c) Prove that $\operatorname{rank}(A^*A) = \operatorname{rank}(A)$.

(d) Prove that the columns of $A$ are linearly independent if and only if $A^*A$ is invertible.

~~proof~~:

(a) If $\vec{v} \in \ker(A)$ then $A^*A = A^*0 = 0$ and thus $\vec{v} \in \ker(A^*A)$.

(b) If $\vec{v} \in \ker(A^*A)$ then $\langle A\vec{v}, A\vec{v}\rangle = \langle \vec{v}, A^*A\vec{v}\rangle = 0$ and thus $A\vec{v} = 0$ which proves that $\vec{v} \in \ker(A)$.

(c) By (a-b) we have that $\ker(A^*A) = \ker(A)$ and thus by rank nullity

$$\operatorname{rank}(A^*A) = n - \dim(\ker(A^*A)) = n - \dim(\ker(A)) = \operatorname{ran}(A).$$

(d) Columns of $A$ are linearly independent $\Leftrightarrow \operatorname{rank}(A) = n \Leftrightarrow \dim(\ker(A)) = 0$ $\Leftrightarrow \dim(\ker(A^*A)) = 0 \Leftrightarrow A^*A$ is invertible.

$\blacksquare$

#6

Let $V = P_\infty(\mathbb{R})$ and $S \subset V$ be defined by $S = \operatorname{span}\{1, t, t^2\}$. On $V$ define the inner product by

$$\langle f, g\rangle = \int_0^\infty f(t)g(t)e^{-t}\,dt.$$

(a) Use integration by parts to show for all $k \in \mathbb{N}$ that

$$\int_0^\infty t^k e^{-t}\,dt = k!$$

(b) With respect to this inner product, prove for all $m, n \in \mathbb{N}$ that $\langle t^m, t^n\rangle = (m+n)!$ and $\|t^m\|^2 = (2m)!$.

(c) With respect to this inner product, find orthonormal functions $\{\vec{q}_0, \vec{q}_1, \vec{q}_2\}$ such that $\operatorname{span}\{q_0, q_1, q_2\} = S$.

Solution:

(a) Let $I(k)=\int_0^\infty t^k e^{-t}dt$. Integrating by parts we have that

$$I(k) = -\cancel{e^{-t}t^k}\Big|_0^\infty + \int_0^\infty k\cdot t^{k-1}e^{-t}dt$$
$$= k\int_0^\infty t^{k-1}e^{-t}dt$$
$$= k\,I(k-1)$$
$$= k\cdot(k-1)\,I(k-2)$$
$$\vdots$$
$$= k(k-1)(k-2)\cdots I(0).$$

Now, $I(0)=\int_0^\infty e^{-t}dt = -e^{-t}\Big|_0^\infty = 1$. Consequently,

$$I(k)=k!$$

(b) Computing, we have

$$\langle t^m, t^n\rangle = \int_0^\infty t^m t^n e^{-t}dt$$
$$= \int_0^\infty t^{(m+n)} e^{-t}dt$$
$$= (m+n)!$$

(c) Computing, we have that

$$\|t^m\|^2 = \langle t^m, t^m\rangle$$
$$= \int_0^\infty t^m t^m e^{-t}dt$$
$$= \int_0^\infty t^{2m} e^{-t}dt$$
$$= (2m)!$$

(d)

1. Since $\|1\| = \|t^0\| = 1$, we set $q_0 = 1$.

2. Let $\vec{w}_1 = t - \langle t, 1\rangle 1$
$$= t-1.$$

Now, $\|\vec{w}_1\|^2 = \langle \vec{w}_1, \vec{w}_1\rangle = \langle t-1, t-1\rangle = \langle t, t\rangle - 2\langle t, 1\rangle + \langle 1, 1\rangle$ and thus

$$\|\vec{w}_1\|^2 = 2! - 2! + 0! = 1.$$

Therefore, we set $q_1 = t-1$.

3. We set

$$\vec{w}_2 = t^2 - \langle t^2, t-1\rangle (t-1) - \langle t^3, 1\rangle 1$$
$$= t^2 - (\langle t^3, t\rangle - \langle t^3, 1\rangle)(t-1) - 2!$$
$$= t^2 - (3! - 2!)(t-1) - 2$$
$$= t^2 - 4(t-1) - 2$$
$$= t^2 - 4t + 2$$

Now,

$$\|\vec{w}_2\|^2 = \langle t^2 - 4t + 2, t^2 - 4t + 2\rangle$$
$$= \langle t^2, t^2\rangle + 2\langle t^2, -4t\rangle + 2\langle t^2, 2\rangle + \langle -4t, -4t\rangle + 2\langle -4t, 2\rangle + \langle 2, 2\rangle$$
$$= \langle t^2, t^2\rangle - 8\langle t^2, t\rangle + 4\langle t^2, 1\rangle + 16\langle t, t\rangle - 16\langle t, 1\rangle + 4\langle 1, 1\rangle$$
$$= 4! - 8\cdot 3! + 4\cdot 2! + 16\cdot 2! - 16\cdot 1! + 4\cdot 0!$$
$$= 24 - 48 + 8 + 32 - 16 + 4$$
$$= -24 + 40 - 12$$
$$= 4$$

Consequently, we set

$$\vec{q}_2 = \frac{1}{2}(t^2 - 4t + 2).$$

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#7

Show $A \in M_{2\times 2}(\mathbb{C})$ is unitary with respect to the standard complex inner product if and only if

$$A = \begin{bmatrix} a & b \\ -\bar{b}e^{i\phi} & \bar{a}e^{i\phi} \end{bmatrix}$$

for some $a, b \in \mathbb{C}$ satisfying $|a|^2 + |b|^2 = 1$ and $\phi \in \mathbb{R}$.

Solution:

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{n \times n}(\mathbb{C})$. Therefore, $A$ is unitary if and only if its columns are orthonormal. Therefore, $A$ is unitary if and only if

$$\bar{a}a + \bar{c}c = 1 \quad (1)$$

$$\bar{a}b + \bar{c}d = 0 \quad (2)$$

$$\bar{b}b + \bar{d}d = 1 \quad (3)$$

From Equation (2) we have that

$$\bar{c} = -\frac{\bar{a}b}{d} \quad \text{and} \quad c = -\frac{a\bar{b}}{\bar{d}}$$

and thus substituting into Equation (1) we have that

$$\bar{a}a + \frac{a\bar{a}b\bar{b}}{d\bar{d}} = 1$$

Now, $\bar{d}d = 1 - \bar{b}b$ and thus

$$\bar{a}a + \frac{a\bar{a}b\bar{b}}{1-\bar{b}b} = 1$$

$$\Rightarrow \bar{a}a - \bar{a}a\bar{b}b + a\bar{a}b\bar{b} = 1 - b\bar{b}$$

$$\Rightarrow \bar{a}a + \bar{b}b = |a|^2 + |b|^2 = 1.$$

From Equations (1) and (3) we have that

$$\bar{c}c = 1 - \bar{a}a = \bar{b}b \quad \text{and} \quad \bar{d}d = 1 - \bar{b}b = \bar{a}a$$

$$\Rightarrow |c|^2 = |b|^2 \quad \text{and} \quad |d|^2 = |a|^2$$

Therefore, there exists $\theta, \phi \in \mathbb{R}$ such that

$$c = -\bar{b}e^{i\theta} \quad \text{and} \quad d = \bar{a}e^{i\phi}$$

Therefore, by equation (2) we have

$$\bar{a}b - b\bar{a}e^{-i\theta}e^{i\phi} = 0$$

Which is only true if $\theta = \phi$. ∎