MTH 225
Homework #1

Due Date: January 22, 2025

1. How should the coefficients $a, b, c \in \mathbb{R}$ be chosen so that the systems $ax + by + cz = 3$, $ax - y + cz = 1$, and $x + by - cz = 2$, has the solution $x = 1$, $y = 2$, and $z = -1$.

2. For what values of x, y, z, w are the matrices

$$A = \begin{bmatrix} x+y & x-z \\ y+w & x+2w \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

   equal?

3. Consider the following system of equations for $x, y, z \in \mathbb{R}$:

$$\begin{aligned} 2 &= ax + bz, \\ b &= ax + 2y + az, \\ a &= bx + 2y + az. \end{aligned}$$

   (a) For what values of $a, b \in \mathbb{R}$ does this system have a unique solution?
   (b) For what values of $a, b \in \mathbb{R}$ does this system have infinitely many solutions?
   (c) For what values of $a, b \in \mathbb{R}$ does this system have no solution?

4. Consider the following matrices $A, B, C \in M_{2\times 2}(\mathbb{R})$:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}.$$

   (a) Show that $AB = AC$ even though $B \neq C$.
   (b) Show that $B^2 - C^2 \neq (B - C)(B + C)$.

5. A matrix $S \in M_{n\times n}(\mathbb{R})$ is said to be a square root of the matrix $A \in m_{n\times n}(\mathbb{R})$ if $S^2 = A$.

   (a) Show that $S = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$ is a square root of the matrix $A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$.
   (b) Find all real square roots of the 2×2 identity matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or prove that none exist.
   (c) Find all real square roots of the matrix $-I = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ or prove that none exist.

6. A matrix $A \in M_{n\times n}(\mathbb{R})$ is called idempotent if $A^2 = A$.

   (a) Show that $A \in M_{n\times n}(\mathbb{R})$ is idempotent if and only if $I - A$ is idempotent, where $I \in M_{n\times n}(\mathbb{R})$ is the $n \times n$ identity matrix.

   (b) Suppose $A, B \in M_{n\times n}(\mathbb{R})$ are idempotent. Show that $(A - B)^3 = A - B$ or explain why it is not true.

   (c) Suppose $A \in M_{n\times n}(\mathbb{R})$ is idempotent. Show that A is invertible if and only if $A = I$.

7. Suppose $A \in M_{n\times n}(\mathbb{R})$ satisfies $A^2 - 3A + I = 0$, where $0 \in M_{n\times n}(\mathbb{R})$ is the $n \times n$ zero matrix and $I \in M_{n\times n}(\mathbb{R})$ is the identity matrix. Show that $A^{-1} = 3I - A$.

8. For $x, y, z, w \in \mathbb{R}$ and $x_1, \dots, x_n \in \mathbb{R}$, the Vandermonde matrices are defined by

$$V_1 = \begin{bmatrix}1\end{bmatrix},\ V_2 = \begin{bmatrix}1 & x\\ 1 & y\end{bmatrix},\ V_3 = \begin{bmatrix}1 & x & x^2\\ 1 & y & y^2\\ 1 & z & z^2\end{bmatrix},\ V_4 = \begin{bmatrix}1 & x & x^2 & x^3\\ 1 & y & y^2 & y^3\\ 1 & z & z^2 & z^3\\ 1 & w & w^2 & w^3\end{bmatrix},\ V_n = \begin{bmatrix}1 & x_1 & x_1^2 & \cdots & x_1^{n-1}\\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1}\\ \vdots & \vdots & \vdots & & \vdots\\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1}\end{bmatrix}.$$

   (a) Compute the determinants of V_1, V_2, V_3, and V_4 and simplify your answers as much as possible. **Hint:** Factor as much as possible.

   (b) Find a general formula for the determinant of V_n. You do not have to formally prove this.

   (c) Find conditions on $x_1, x_2, \dots, x_n$ that are necessary and sufficient for V_n to be invertible.