MTH 225
Homework #2

Due Date: January 29, 2025

1. If $z = 1 + 2i$, $w = 2 - i$, and $\zeta = 4 + 3i$, write the following complex expressions in the form $a + bi$, where $a, b \in \mathbb{R}$.
   (a) $z + 3w$
   (b) $-2w + \overline{\zeta}$
   (c) z^2
   (d) $w^3 + w$
   (e) $\text{Im}(\zeta^{-1})$
   (f) w/z
   (g) $\zeta^2 + 2\overline{\zeta} + 3$
2. Write the given complex number in the form $a + bi$, where $a, b \in \mathbb{R}$.
   (a) i^{101}
   (b) $\frac{1+2i}{5-3i}$
   (c) $e^{i\pi/6}$
   (d) $(1 + i\sqrt{3})^{10}$
3. Write the given complex number in the form $a + bi$, where $a, b \in \mathbb{R}$.
   (a) $e^{-i\pi/2}$
   (b) $\dfrac{e^{1+3\pi i}}{e^{-1+\pi/2i}}$
   (c) $\frac{e^{3i}-e^{-3i}}{2i}$
   (d) e^{e^i}
4. Let $z \in \mathbb{C}$ and assume $z \neq 0$. Prove the following
   (a) $|\text{Re}(z)| \leq |z|$ and $|\text{Im}(z)| \leq |z|$.
   (b) $\text{Re}(z) = (z + \overline{z})/2$ and $\text{Im}(z) = -i(z - \overline{z})/2$.
   (c) $|z| = 1$ if and only if $1/z = \overline{z}$.
   (d) If $|z| = 1$ and $z \neq 1$, then $\text{Re}((1 - z)^{-1}) = 1/2$.

5. Compute the following
   (a) $5 + 3^4$ in $\mathbb{Z}_{11}$.
   (b) 7^{-1} in $\mathbb{Z}_{13}$.
   (c) 3^{101} in $\mathbb{Z}_5$.
   (d) -16 in $\mathbb{Z}_{19}$.
   (e) $7 \cdot (3 + 13)$ in $\mathbb{Z}_{11}$.
   (f) 8^{-1} in $\mathbb{Z}_{15}$.

6. Find the inverse of the following matrix:
$$\begin{bmatrix} 2-3i & 4 \\ 1 & 2 \end{bmatrix}.$$

7. Show that 10 does not have a multiplicative inverse in $\mathbb{Z}_{15}$.

8. Find all solutions to the equation $x^2 = 1$ in $\mathbb{Z}_8$.

9. Find a quadratic polynomial $x^2 + bx + c$ over $\mathbb{Z}_6$ which has four distinct roots in $\mathbb{Z}_6$.

10. Find two nonzero elements in $\mathbb{Z}_{14}$ whose product is 0.

11. Solve the following system of linear equations over $\mathbb{Z}_3$:
$$\begin{aligned} 2x + y &= 1, \\ 2x + 2y &= 2. \end{aligned}$$

12. Find all solutions to the following system of linear equations over $\mathbb{Z}_5$:
$$\begin{aligned} 2w + 2x + y + z &= 4, \\ w + x + 3y + z &= 1, \\ x + y &= 1. \end{aligned}$$
Remember, in $\mathbb{Z}_5$ you should only have a finite number of solutions.

13. Find the inverse of the following matrix over $\mathbb{Z}_3$:
$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$