MTH 225
Homework #3

Due Date: February 05, 2025

1. Let $Q = \{(x, y) : x, y > 0\} \subset \mathbb{R}^2$ denote the positive quadrant in the real plane.
   (a) Show that Q is not a vector space over $\mathbb{R}$ under the standard rules for vector addition and scalar multiplication.
   (b) Show that Q is a vector space over $\mathbb{R}$ if we define vector addition by
   $$(x_1, y_1) + (x_2, y_2) = (x_1x_2, y_1y_2)$$
   and scalar multiplication by
   $$c(x, y) = (x^c, y^c).$$
   Hint: You will have to show all of the properties of a vector space are true. You will also have to determine the zero vector $\mathbf{0}$.

2. Let $V \subset M_{2\times 2}(\mathbb{C})$ be the set of complex matrices of the form
   $$\mathbf{v} = \begin{bmatrix} 1 & v \\ 0 & 1 \end{bmatrix}$$
   for some $v \in \mathbb{C}$. Define vector addition and scalar multiplication over the field $F = \mathbb{C}$ by
   $$\mathbf{v} + \mathbf{w} = \begin{bmatrix} 1 & v \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & w \\ 0 & 1 \end{bmatrix} \text{ (ordinary matrix multiplication)}$$
   and
   $$c\mathbf{v} = \begin{bmatrix} 1 & cv \\ 0 & 1 \end{bmatrix}.$$
   Prove that V is a vector space over $\mathbb{C}$. **Hint**: You will have to show all of the properties of a vector space are true. You will also have to determine the zero vector $\mathbf{0}$.

3. Consider the vector space $V = \mathbb{R}^2$.
   (a) Construct an example of a subset $S \subset V$ in which for all $c \in \mathbb{R}$ and $\mathbf{u} \in S$, $c\mathbf{u} \in S$ but S is not a subspace of V.
   (b) Construct an example of a subset $S \subset V$ in which for all $\mathbf{u}, \mathbf{v} \in V$, $\mathbf{u} + \mathbf{v} \in S$ but S is not a subspace of V.

4. Determine which of the following sets of vectors $\mathbf{u} = (u_1, \ldots, u_n)^T$ are subspaces of $\mathbb{C}^n$. If a set is a subspace, prove it, if not, present a counterexample.

   (a) $A = \{\mathbf{u} \in \mathbb{C}^n : u_1 = \cdots = u_n\}$

   (b) $B = \{\mathbf{u} \in \mathbb{C}^n : \mathrm{Re}(u_i) \geq 0\}$

   (c) $C = \{\mathbf{u} \in \mathbb{C}^n : u_1 = u_n = 0\}$

   (d) $D = \{\mathbf{u} \in \mathbb{C}^n : u_1 - u_n = 1\}$

5. Determine which of the following are vector spaces. If a set with the given operations is a vector space, prove it, if not, provide a counterexample. **Hint:** You can show something is a vector space by proving it is a subspace of one of the known vector spaces.

   (a) $A = \{f \in C^1(\mathbb{R}) : f(x+1) = f(x)\}$ with the standard operations of addition and scalar multiplication of functions.

   (b) $B = \{f \in C^1(\mathbb{R}) : f(x) \geq 0\}$ with the standard operations of addition and scalar multiplication of functions.

   (c) $C = \{p \in P_n(\mathbb{R}) : p(x) = p(-x)$ with the standard operations of addition and scalar multiplication of polynomials.

   (d) $D = \{p \in P_\infty(\mathbb{R}) : p(x)$ has a factor of $x - 1\}$ with the standard operations of addition and scalar multiplication of polynomials.

6. Determine which of the following sets are subspaces of $C^1(\mathbb{R})$, the vector space of all differentiable functions with continuous derivatives. If a set is a subspace, prove it, if not, present a counterexample.

   (a) $A = \{f \in C^1(\mathbb{R}) : f(2) = f(3)\}$

   (b) $B = \{f \in C^1(\mathbb{R}) : f'(2) = f(3)\}$

   (c) $C = \{f \in C^1(\mathbb{R}) : f'(x) + f(x) = 0\}$

   (d) $D = \{f \in C^1(\mathbb{R}) : f(2 - x) = f(x)\}$

   (e) $E = \{f \in C^1(\mathbb{R}) : f(-x) = e^x f(x)\}$

   (f) $F = \{f \in C^1(\mathbb{R}) : f(x) = a + b|x| \text{ for some } a, b \in \mathbb{R}\}$

7. Prove that the set of functions $u(x)$ that satisfy the following differential equation

$$\frac{d^2u}{dx^2} = xu(x)$$

   forms a subspace of $C^2(\mathbb{R})$.

8. Let $\mathcal{A} \in M_{3\times3}(\mathbb{C})$. Determine which of the following subsets of $M_{3\times3}(\mathbb{C})$ are subspaces of $M_{3\times3}(\mathbb{C})$. If a set is a subspace, prove it, if not, present a counterexample.

   (a) $A = \{X \in M_{3\times3}(\mathbb{C}) : X \text{ is invertible}\}$.

   (b) $B = \{X \in M_{3\times3}(\mathbb{C}) : X \text{ is not invertible}\}$.

   (c) $C = \{X \in M_{3\times3}(\mathbb{C}) : X \text{ is lower triangular}\}$.

   (d) $D = \{X \in M_{3\times3}(\mathbb{C}) : \mathcal{A}X + X^T\mathcal{A} = 0\}$, where X^T denotes the transpose of a matrix.