MTH 225
Homework #4

Due Date: February 12, 2025

1. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}$ be vectors in a vector space V. Prove or provide a counterexample: if $\mathbf{z}$ is a linear combination of $\mathbf{u}, \mathbf{v}, \mathbf{w}$ then $\mathbf{w}$ is a linear combination of $\mathbf{u}, \mathbf{v}, \mathbf{z}$.

2. Suppose $\mathbf{f}(t)$ and $\mathbf{g}(t)$ are functions from $\mathbb{R}$ to $\mathbb{R}^2$.

   (a) Prove that if $\mathbf{f}(t_0)$ and $\mathbf{g}(t_0)$ are linearly independent as vectors in $\mathbb{R}^2$ for some $t_0 \in \mathbb{R}^2$ then $\mathbf{f}(t)$ and $\mathbf{g}(t)$ are linearly independent functions.

   (b) Show that $\mathbf{f}(t) = \begin{bmatrix} 1 \\ t \end{bmatrix}$ and $\mathbf{g}(t) = \begin{bmatrix} 2t-1 \\ 2t^2 - t \end{bmatrix}$ are linearly independent functions.

   (c) Show that $\mathbf{f}(t_0) = \begin{bmatrix} 1 \\ t_0 \end{bmatrix}$ and $\mathbf{g}(t_0) = \begin{bmatrix} 2t_0-1 \\ 2t_0^2 - t_0 \end{bmatrix}$ are linearly dependent for all $t_0 \in \mathbb{R}$.

   Parts (b) and (c) show that the converse of (a) is not true in general.

3. The Wronskian of a pair of differentiable functions $f(x)$, $g(x)$ defined by

$$W[f(x), g(x)] = \det\left(\begin{bmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{bmatrix}\right).$$

   (a) Prove that if f and g are linearly dependent then $W[f(x), g(x)] = 0$.

   (b) Use part (a) to prove that if $W[f(x), g(x)] \neq 0$ then f and g are linearly independent.

   (c) Let $f(x) = x^3$ and $g(x) = |x|^3$. Show that f and g are linearly independent but $W[f(x), g(x)] = 0$. Why does this not contradict parts (a) and (b)?

4. Consider the set $\mathcal{A} = \{1 + x^2, x + x^2, 1 + 2x + x^2\}$.

   (a) Prove that $\mathcal{A}$ is a basis for $P_2(\mathbb{R})$.

   (b) Find the coordinates of $p(x) = 1 + 4x + 7x^2$ in this basis.

5. Let $S_{n\times n}(\mathbb{R})$ denote the subspace of $M_{2\times 2}(\mathbb{R})$ consisting of all symmetric $n \times n$ matrices with complex entries.

   (a) Show that a basis for $S_{2\times 2}(\mathbb{R})$ is given by

$$\mathcal{A} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}.$$

   (b) Find a basis for $S_{3\times 3}(\mathbb{R})$ and determine $\dim(S_{3\times 3}(\mathbb{R}))$. You do not have to prove that your choice is a basis.

6. A real valued matrix is said to be a *semi-magic square* if its row sums and column sums, i.e, the sum of entries in an individual row or column, all add up to the sum number. An example is

$$M = \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix}.$$

(a) Prove that S, the set of all 3×3 semi-magic squares, is a subspace of $M_{3\times 3}(\mathbb{R})$.

(b) Find a basis for S. You do not have to prove that your choice is a basis and, as a hint, $\dim(S) = 6$.

7. Find a linear transformation $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$ such that

$$T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \text{ and } T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

8. Consider the map $T : P_3(\mathbb{R}) \mapsto \mathbb{R}^4$ defined by

$$T(a_3x^3 + a_2x^2 + a_1 + a_0) = \begin{bmatrix} a_1 \\ -a_1 \\ a_2 \\ -a_2 \end{bmatrix}.$$

(a) Show that T is a linear transformation.

(b) Find bases for $\ker(T)$ and $\operatorname{im}(T)$.

9. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ be 2×2 real valued matrices. For each of the following functions, prove that $T : M_{2\times 2}(\mathbb{R}) \mapsto M_{2\times 2}(\mathbb{R})$ is a linear transformation and then find its 4×4 matrix representation with respect to the standard basis:

$$\mathcal{A} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

(a) $T[X] = AX$.

(b) $R[X] = XB$.

(c) $K[X] = AXB$.