MTH 225
Homework #5

Due Date: February 26, 2025

1. The set of vectors $\mathcal{A} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ and $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$ are a basis for $\mathbb{R}^2$.
   (a) Compute $[I(\mathcal{A}, \mathcal{B})]$ and $[I(\mathcal{B}, \mathcal{A})]$.
   (b) Show that $[I(\mathcal{A}, \mathcal{B})]$ and $[I(\mathcal{B}, \mathcal{A})]$ are inverses of each other.
2. Let $V = M_{2\times 2}(\mathbb{C})$ and $W = M_{n\times n}(\mathbb{C})$.
   (a) Find a standard basis for V. You don't need to prove anything, you can just list the matrices. **Hint:** $\dim(V) = 8$.
   (b) What is $\dim(W)$?
   (c) Let $\mathrm{RE} : V \mapsto M_{2\times 2}(\mathbb{R})$ be defined by $\mathrm{RE}(A)$ is the real valued matrix with components $(\mathrm{RE}(A))_{ij} = \mathrm{Re}(A_{ij})$. Show that RE is a linear transformation.
   (d) Find the matrix representation of RE with respect to the standard basis for V and the standard basis for $M_{2\times 2}(\mathbb{R})$.
   (e) Find a basis for $\ker(\mathrm{RE})$ and $\mathrm{im}(A)$. **Hint:** You don't have to do any row reduction or set up any system of linear equations.
3. For $A \in M_{n\times n}(\mathbb{C})$, with entries $A_{ij} \in \mathbb{C}$, the trace function $\mathrm{tr} : M_{n\times n}(\mathbb{C}) \mapsto \mathbb{C}$ is defined by
$$\mathrm{tr}(A) = \sum_{i=1}^{n} A_{ii}.$$
   (a) Show that tr is a linear transformation.
   (b) Explain why $\dim(\mathrm{im}(\mathrm{tr})) = 2$.
   (c) Determine $\dim(\ker(\mathrm{tr}))$.
   (d) Show that if $A, B \in M_{n\times n}(\mathbb{C})$ then $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. **Hint:** Remember that if $C = AB$, then the entries of C are given by $C_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}$.
4. Let $A, B \in M_{n\times n}(\mathbb{C})$. Recall that A and B are similar if there exists a matrix $P \in M_{n\times n}(\mathbb{C})$ such that $A = PBP^{-1}$.
   (a) Show that if A is similar to B, then $A - \lambda I$ is similar to $B - \lambda I$ for every $\lambda \in \mathbb{C}$.
   (b) Show that if there exists a $\lambda \in \mathbb{C}$ such that $A - \lambda I$ is similar to $B - \lambda I$, then A is similar to B.
   (c) Show that if A is similar to B then $\mathrm{tr}(A) = \mathrm{tr}(B)$.
   (d) Show that if A is similar to B then $\det(A) = \det B$.

5. Recall that a relationship on a set X, denoted $\sim$, is called an equivalence relationship if the following properties are satisfied

   (a) **Reflexivity:** If $a \in X$ then $a \sim a$.

   (b) **Symmetry:** If $a, b \in X$ then $a \sim b$ if and only if $b \sim a$.

   (c) **Transitivity:** If $a, b, c \in X$ and $a \sim b$ and $b \sim c$ then $a \sim c$.

   Show that on $M_{n\times n}(\mathbb{C})$ similarity between matrices is an equivalence relationship.

6. Let $A \in M_{n\times n}(\mathbb{C})$.

   (a) Show that A is similar to I if and only if $A = I$.

   (b) Show that A is similar to 0 if and only if $A = 0$.

7. Let $A \in M_{n\times n}(\mathbb{C})$.

   (a) Show that the linear system $A\mathbf{x} = \mathbf{y}$ has a unique solution for some $\mathbf{y} \in \mathbb{C}^n$ if and only if it has a solution for all $\mathbf{y} \in \mathbb{C}^n$.

   (b) Show that the linear system $A\mathbf{x} = \mathbf{y}$ has a unique solution for all $\mathbf{y} \in \mathbb{C}^n$ if and only if the only solution to $A\mathbf{x} = 0$ is $\mathbf{x} = 0$.