MTH 225
Homework #6

Due Date: March 05, 2025

1. Suppose $A \in M_{n\times n}(\mathbb{R})$ has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 3$ with corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Find the matrix representation of the linear transformation $L[\mathbf{v}] = A\mathbf{u}$ with respect to following bases.
   (a) The standard basis.
   (b) The basis of eigenvectors.
   (c) The basis $\mathcal{A} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$.
2. Let $\mathbf{u} \in \mathbb{R}^n$ and $\mathbf{v} \in \mathbb{R}^n$ and $A = \mathbf{u}\mathbf{v}^T$.
   (a) Find the columns of A in terms of $\mathbf{u}$ and $\mathbf{v}$.
   (b) Show that $\text{rank}(A) = 1$.
3. Suppose $P \in M_{n\times n}(\mathbb{C})$ satisfies $P^2 = P$. Find all of the possible eigenvalues of P.
4. Suppose $N \in M_{n\times n}(\mathbb{C})$ satisfies $N^k = 0$ for some $k \in \mathbb{N}$. Find all possible eigenvalues of N.
5. Suppose $A \in M_{n\times n}(\mathbb{C})$ satisfies $A^4 = I$. Find all possible eigenvalues of A.
6. Suppose $A \in M_{3\times 3}(\mathbb{R})$ has $\lambda = 1$ as its only eigenvalue.
   (a) What is the algebraic multiplicity of λ?
   (b) Find an example of such a matrix A in which the geometric multiplicity of A is 1.
   (c) Find an example of such a matrix A in which the geometric multiplicity of A is 2.
   (d) Find an example of such a matrix A in which the geometric multiplicity of A is 3
7. Suppose $T : V \mapsto V$ is an invertible linear transformation and suppose $\lambda \neq 0$ is an eigenvalue with corresponding eigenvector $\mathbf{v}$.
   (a) Prove that $\lambda \neq 0$.
   (b) Prove that $\frac{1}{\lambda}$ is an eigenvalue of T^{-1} with corresponding eigenvector $\mathbf{v}$.
8. Suppose that $A \in M_{n\times n}(\mathbb{C})$ is diagonalizable.
   (a) Show that $cA + dI$, where c, d are any scalars, is diagonalizable.
   (b) Show that A^2 is diagonlizable.
   (c) Give an example of a matrix A that is not diagonalizable but A^2 is.

9. Suppose $A, B \in M_{n\times n}(\mathbb{C})$ are diagonlizable matrices with the same eigenspaces (but not necessarily the same eigenvalues). Prove that $AB = BA$.

10. Suppose $A \in M_{n\times n}(\mathbb{C})$ is a diagonlizable matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$.
    (a) Prove that $\det(A) = \lambda_1 \cdots \lambda_n$.
    (b) Prove that $\mathrm{tr}(A) = \lambda_1 + \ldots + \lambda_n$.
    (c) If $A \in M_{2\times 2}(\mathbb{C})$ is a diagonlizable, show that its eigenvalues λ_1, λ_2 satisfy

$$\lambda_{1,2} = \frac{1}{2}\left(\mathrm{tr}(A) \pm \sqrt{\mathrm{tr}(A)^2 - 4\det(A)}\right).$$

Hint: You can use results from previous homeworks.

11. Suppose $T : P_4(\mathbb{R}) \mapsto P_4(\mathbb{R})$ is defined by $T(p(x)) = x\frac{dp}{dx}$.
    (a) Show that T is a linear transformation.
    (b) Find a basis for $\ker(T)$.
    (c) Find a basis for $\mathrm{im}(T)$.
    (d) Find all eigenvectors and eigenvalues of T. **Hint:** Don't try to do this using matrices.
    (e) Find the matrix representation of T with respect to the standard basis $\mathcal{S} = \{1, x, x^2, x^3, x^4\}$.

12. Define the shift map $S : \mathbb{C}^n \mapsto \mathbb{C}^n$ by

$$S\left(\left[a_1, a_2, \ldots, a_{n-1}, a_n\right]^T\right) = \left[a_2, a_3, \ldots, a_n, a_1\right]^T.$$

    (a) Prove that S is a linear transformation.
    (b) Prove that the vectors $\omega_0, \omega_1, \ldots, \omega_n$ defined by

$$\omega_k = \left[1, e^{2k\pi i/n}, e^{4k\pi i/n}, \ldots, e^{2(n-1)k\pi i/n}\right]^T,$$

    are eigenvectors of S. What are the corresponding eigenvalues?
    (c) Find the matrix representation of S with respect to the standard basis.

13. Let $T : \mathbb{R}^\infty \mapsto \mathbb{R}^\infty$ and $S : \mathbb{R}^\infty \mapsto \mathbb{R}^\infty$ be defined by

$$T(a_0, a_1, a_2, \ldots) = (0, a_0, a_1, a_2, \ldots) \text{ and } S(a_0, a_1, a_2, \ldots) = (a_1, a_2, a_3, \ldots).$$

    (a) Show that T and S are linear transformations.
    (b) Show that $T \circ S$ is the identity transformation but $S \circ T$ is not.
    (c) Show that T has no eigenvalues.
    (d) Find all of the eigenvalues and eigenvectors of S.