MTH 225
Homework #7

Due Date: March 19, 2025

1. Prove that if $b > 1$ then the function $\langle \cdot, \cdot \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \mapsto \mathbb{R}$ defined by

$$\langle \mathbf{v}, \mathbf{w} \rangle = v_1 w_1 - v_1 w_2 - v_2 w_1 + b v_2 w_2$$

   is an inner product on $\mathbb{R}^2$.

2. Show that the function $\langle \cdot, \cdot \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \mapsto \mathbb{R}$ defined by

$$\langle \mathbf{v}, \mathbf{w} \rangle = v_1 w_1 + v_1 w_2 + v_2 w_1 + v_2 w_2$$

   does not define an inner product on $\mathbb{R}^2$.

3. Let V be a vector space with a complex inner product $\langle \cdot, \cdot \rangle : V \times V \mapsto \mathbb{C}$.
   (a) Prove that $\langle \mathbf{x}, \mathbf{v} \rangle = 0$ for all $\mathbf{v} \in V$ if and only if $\mathbf{x} = 0$.
   (b) Prove that $\langle \mathbf{x}, \mathbf{v} \rangle = \langle \mathbf{y}, \mathbf{v} \rangle$ for all $\mathbf{v} \in V$ if and only if $\mathbf{x} = \mathbf{y}$.
   (c) Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for V. Prove that $\langle \mathbf{x}, \mathbf{v}_i \rangle = \langle \mathbf{y}, \mathbf{v}_i \rangle$, $i = 1, \ldots, n$, if and only if $\mathbf{x} = \mathbf{y}$.
   (d) Prove that $\mathbf{x} \in \mathbb{R}^n$ solves the linear system $A\mathbf{x} = \mathbf{b}$ if and only if

$$\mathbf{x}^T A^T \mathbf{v} = \mathbf{b^T} \mathbf{v}$$

   for all $\mathbf{v}$.

4. Let $V = M_{n \times n}(\mathbb{R})$. Prove that

$$\langle A, B \rangle = \mathrm{tr}(A^T B)$$

   defines an inner product on V.

5. Let $V = C^1([0, 1])$. Let $\langle \cdot, \cdot \rangle : V \times V \mapsto \mathbb{R}$ be defined by

$$\langle f, g \rangle = \int_0^1 \left(f(x)g(x) + f'(x)g'(x) \right) dx.$$

   (a) Prove that $\langle \cdot, \cdot \rangle$ defines an inner product on V.
   (b) Write out an expression for the norm corresponding to this inner product.

6. Let $V = C^1([0,1])$ and $W = \{f \in C^1([0,1]) : f(0) = 0\}$. Let $\langle \cdot, \cdot \rangle : V \times V \mapsto \mathbb{R}$ be defined by

$$\langle f, g \rangle = \int_0^1 f'(x) g'(x) dx.$$

   (a) Prove that $\langle \cdot, \cdot \rangle$ is not an inner product on V.

   (b) Prove that $\langle \cdot, \cdot \rangle$ is an inner product on W.

7. Let V be a complex inner product space with inner product $\langle \cdot, \cdot \rangle : V \times V \mapsto \mathbb{C}$.

   (a) If $z \in \mathbb{C}$, prove that $z + \overline{z} = 2\mathrm{Re}(z)$ and $z - \overline{z} = 2i\mathrm{Im}(z)$.

   (b) Prove for all $\mathbf{v}, \mathbf{w} \in V$ that

$$\|\mathbf{v} + \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + 2\mathrm{Re}(\langle \mathbf{v}, \mathbf{w} \rangle) + \|\mathbf{w}\|^2.$$

   (c) Prove for all $\mathbf{v}, \mathbf{w} \in V$ that

$$\|\mathbf{v} - i\mathbf{w}\|^2 = \|\mathbf{v}\|^2 + 2\mathrm{Im}(\langle \mathbf{v}, \mathbf{w} \rangle) + \|\mathbf{w}\|^2.$$

   (d) Prove for all $\mathbf{v}, \mathbf{w} \in V$ that

$$\langle \mathbf{v}, \mathbf{w} \rangle = \frac{1}{4} \left(\|\mathbf{v} + \mathbf{w}\|^2 - \|\mathbf{v} - \mathbf{w}\|^2 - i\|\mathbf{v} + i\mathbf{w}\|^2 + i\|\mathbf{v} - i\mathbf{w}\|^2 \right).$$

8. Let $\mathbf{v}, \mathbf{w} \in V$ and suppose $\langle \cdot, \cdot \rangle$ is a real inner product on V. Prove that

$$\|\mathbf{v} + \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2$$

   if and only if $\mathbf{v}$ and $\mathbf{w}$ are orthogonal. How does this result change if $\langle \cdot, \cdot \rangle$ is a complex inner product?

9. Let $\mathbf{v}, \mathbf{w} \in V$ be a basis for V and suppose that $\langle \cdot, \cdot \rangle$ is an inner product on V. Suppose further that $\|\mathbf{v}\| = 1$ and $\|\mathbf{w}\| = 1$.

   (a) If $\langle \cdot, \cdot \rangle$ is a real-valued inner product, prove that $\mathbf{v} + \mathbf{w}$ and $\mathbf{v} - \mathbf{w}$ forms an orthogonal basis for V.

   (b) How does this result change if $\langle \cdot, \cdot \rangle$ is a complex-valued inner product?

10. Prove that the polynomials $p_0(x) = 1$, $p_1(x) = x$, $p_2(x) = x^2 - \frac{1}{3}$, and $p_3(x) = x^3 - \frac{3}{5}x$ form an orthogonal basis for $P_2(\mathbb{R})$ with respect to the following inner product:

$$\langle f, g \rangle = \int_{-1}^1 f(x) g(x) dx.$$

11. Show for $k \in \mathbb{N}$ that the complex exponentials e^{ikx} are orthogonal under the inner product defined by

$$\langle f, g \rangle = \int_{-\pi}^{\pi} \overline{f(x)} g(x) dx.$$