# MTH 225
# Homework #8

Due Date: March 26, 2025

1. Consider the following vectors

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix}, \mathbf{u}_2 = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\1\end{bmatrix}, \mathbf{u}_3 = \frac{1}{\sqrt{6}}\begin{bmatrix}-1\\1\\2\end{bmatrix}.$$

   (a) Show that $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ forms an orthonormal basis for $\mathbb{R}^3$ with respect to the standard inner product.

   (b) Find the coordinates of $\mathbf{v} = \begin{bmatrix}2 & 1 & 3\end{bmatrix}^T$ with respect to the above basis. **Hint:** Do not solve a $3 \times 3$ system of equations.

2. Let $\mathbf{u}_1, \ldots, \mathbf{u}_n$ be an orthonormal system in $\mathbb{C}^n$ with respect to the standard complex inner product. If $\mathbf{v} \in \mathbb{C}^n$, prove that

$$\|\mathbf{v}\|^2 = \sum_{i=1}^{n} |\langle \mathbf{v}, \mathbf{u}_i \rangle|^2,$$

   where $|\cdot|$ denotes the complex modulus.

3. Let $A \in M_{n\times n}(\mathbb{C})$. In these problems we assume the standard complex inner product on $\mathbb{C}^n$.

   (a) Prove for all $\mathbf{v}, \mathbf{w} \in \mathbb{C}^n$ that $\langle A\mathbf{v}, \mathbf{w}\rangle = \langle \mathbf{v}, A^*\mathbf{w}\rangle$.

   (b) Prove that $\ker(A^*) = (\mathrm{im}(A))^\perp$.

   (c) Prove that $\mathrm{im}(A^*) = (\ker(A))^\perp$

   (d) Prove that $\ker(A) = (\mathrm{im}(A^*))^\perp$

   (e) Prove that $\mathrm{im}(A) = (\ker(A^*))^\perp$.

   (f) Let $\mathbf{b} \in \mathbb{C}^n$. Prove there exists $\mathbf{v} \in \mathbb{C}^n$ that satisfies $A\mathbf{v} = \mathbf{b}$ if and only if $\langle \mathbf{b}, \mathbf{w}\rangle = 0$ for all $\mathbf{w}$ satisfying $A^*\mathbf{w} = 0$.

4. Let $A \in M_{n\times n}(\mathbb{C})$. In these problems we assume the standard complex inner product on $\mathbb{C}^n$.

   (a) Prove that $\ker(A) \subseteq \ker(A^*A)$

   (b) Prove that $\ker(A^*A) \subset \ker(A)$. **Hint:** If $\mathbf{v} \in \ker(A^*A)$, what is $\langle A\mathbf{v}, A\mathbf{v}\rangle$?

   (c) Prove that $\mathrm{rank}(A^*A) = \mathrm{rank}(A)$.

   (d) Prove that the columns of $A$ are linearly independent if and only if $A^*A$ is invertible.

5. Find an orthonormal basis, with respect to the standard inner product, for the following subspaces of $\mathbb{R}^4$:

   (a) The span of the vectors

$$\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 2 \\ 1 \end{bmatrix},$$

   .

   (b) The kernel of the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 & -1 \\ 3 & 2 & -1 & -1 \end{bmatrix}.$$

   (c) The image of the matrix

$$B = \begin{bmatrix} 1 & -2 & 2 \\ 2 & -4 & 1 \\ 0 & 0 & -1 \\ -2 & 4 & 5 \end{bmatrix}.$$

   (d) The set of all vectors orthogonal to the vector

$$\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}.$$

6. Let $V = P_\infty(\mathbb{R})$ be the vector space of polynomials of arbitrary large degree and $S \subset V$ be defined by $S = \text{span}\{1, t, t^2\}$. On $V$ define the inner product $\langle \cdot, \cdot \rangle$ by

$$\langle f, g \rangle = \int_0^\infty f(t)g(t)e^{-t}dt.$$

   (a) Use integration by parts to show that for all $k \in \mathbb{N}$ that

$$\int_0^\infty t^k e^{-t} dt = k!.$$

   (b) With respect to this inner product, prove that for $m, m \in \mathbb{N}$ that $\langle t^m, t^n \rangle = (m+n)!$ and $\|t^m\|^2 = (2m)!$.

   (c) With respect to this inner product, find a set of orthonormal functions $\{q_0, q_1, q_2\}$ such that $\text{span}\{q_0, q_1, q_2\} = S$.

7. Show $A \in M_{2\times 2}(\mathbb{C})$ is unitary with respect to the standard complex inner product if and only if

$$A = \begin{bmatrix} a & b \\ -\overline{b}e^{i\phi} & \overline{a}e^{i\phi} \end{bmatrix},$$

   for some $a, b \in \mathbb{C}$ satisfying $|a|^2 + |b|^2 = 1$ and $\phi \in \mathbb{R}$.

8. Suppose $A \in M_{n\times n}(\mathbb{C})$ is a unitary matrix with respect to the standard complex inner product. Prove that $A\mathbf{v} = \mathbf{b}$ has a solution if and only if $\mathbf{b} \in (\ker(A))^\perp$.