Lecture 1: Solving Linear Systems

Example:

Solve

$$3x+5y-4z=7$$
$$-3x-2y+4z=-1$$
$$6x+y-8z=-4$$

Idea: The coefficents are what really matter and can be be put into an augmented matrix.

$$\left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{array}\right] \begin{array}{l} \\ +R1 \\ -2R1 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ 0 & 3 & 0 & 6 \\ 0 & -9 & 0 & -18 \end{array}\right] \begin{array}{l} \\ /3 \\ +3R2 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} /3 \\ \\ \\ \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 5/3 & -4/3 & 7/3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} z = \text{any real number} \\ y=2 \\ x+\frac{5}{3}\cdot 2-\frac{4}{3}z=\frac{7}{3} \end{array} \Rightarrow \begin{array}{l} z\in\mathbb{R} \\ y=2 \\ x=-1+\frac{4}{3}z \end{array}$$

The solution can be written in the form:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1+\frac{4}{3}t \\ 2 \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix},$$

where $t\in\mathbb{R}$.

We can also use set notation to describe the set of solutions:

$$V=\left\{\begin{bmatrix} x \\ y \\ z \end{bmatrix}\in\mathbb{R}^3 : x=-1+\tfrac{4}{3}z \text{ and } y=2\right\}$$

We can also write this in the form:

$$V=\left\{\vec{v}\in\mathbb{R}^3 : \vec{v}-\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}\in \operatorname{span}\left\{\begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}\right\}\right\}.$$

Comments:

1. The process of row reduction required the following things
   (i) A linear set of equations.
   (ii) A number system in which addition, inverse addition (subtraction), multiplication, inverse multiplication (division), all make sense.
2. Describing solutions compactly used the idea of span which is an example of a vector space.

Example:

Linear algebra is not just used to solve equations involving real numbers. As an example, computer screens emit colors which are a mixture of red, blue, and green. In 16 bit color, each pixel has 2^{16} variations of red, blue, and green. This means there are

$$2^{16} \cdot 2^{16} \cdot 2^{16} = 2^{48} \text{ (trillions of possible colors)}$$

Instead of defining a name for each color, we take a linear combination of them

$$\text{color} = c_1 \cdot \text{red} + c_2 \cdot \text{blue} + c_3 \cdot \text{green}$$
$$= c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_{B_1} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_{B_1} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_{B_1} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}_{B_1} \qquad B_1 = \text{primary basis}$$

However, when printing the basis of colors is magenta, cyan, yellow

$$\text{color} = d_1 \cdot \text{magenta} + d_2 \cdot \text{cyan} + d_3 \cdot \text{yellow}$$
$$= d_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_{B_2} + d_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_{B_2} + d_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_{B_2} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}_{B_2} \qquad B_2 = \text{printer basis}$$

How can a computer transform between these two representations of color? To do so, we will need to learn about the idea of a basis and linear transformations.

Example:

Linear algebra can be used to determine when solutions exist. For example, consider the system

$$\begin{aligned} x+y+z&=4 \\ y&=2 \\ (a^2-4)z&=a-2 \end{aligned} \Rightarrow \left[\begin{array}{ccc:c} 1 & 1 & 1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & a^2-4 & a-2 \end{array}\right]$$

where $a \in \mathbb{R}$.

1. If $a \neq \pm 2$ then

$$z=\frac{1}{a+2},\ y=2,\ x=2-\frac{1}{a+2}$$

$$V=\left\{\vec{v}\in\mathbb{R}^3 : \vec{v}=\frac{1}{a+2}\begin{bmatrix} 2a+3 \\ 2a+4 \\ 1 \end{bmatrix}\right\}$$

$\Rightarrow$ Unique solution

2. If $a=2$ we have

$$\begin{aligned} 0\cdot z&=0 \\ y&=2 \\ x&=2-z \end{aligned}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 2-z \\ 2 \\ z \end{bmatrix}=\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}+z\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$V=\left\{\vec{v}\in\mathbb{R}^3 : \vec{v}=\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}+t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \text{ for some } t\in\mathbb{R}\right\}$$

$\Rightarrow$ Infinitely many solutions

3. If $a=-2$ we have

$$\begin{aligned} 0\cdot z&=-4 \\ y&=2 \\ x+y+z&=4 \end{aligned}$$

$\Rightarrow$ Solutions $=\emptyset$ $\Rightarrow$ No solutions