Lecture #10: Conditions for Diagonalization

Algebraic Multiplicity:

If $T: V \to V$ has a matrix representation A, the characteristic polynomial is

$$p(\lambda) = \det(A - \lambda I).$$

The roots of $p(\lambda)$ are the eigenvalues $\lambda_1, \dots, \lambda_n$. The algebraic multiplicity $m_a(\lambda)$ is the number of times λ is a root of the characteristic polynomial.

Example:

1. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \lambda_1 = 1,\ m_a(1) = 2.$

2. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \Rightarrow \lambda_1 = 1,\ m_a(1) = 2$

3. $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \Rightarrow \lambda_1 = 1,\ \lambda_2 = 3,\ m_a(1) = 1,\ m_a(3) = 1.$

Geometric Multiplicity

If $T: V \to V$ has a matrix representation A with eigenvalues $\lambda_1, \dots, \lambda_n$ then

E_{λ_i} = span of eigenvectors associated with λ_i.

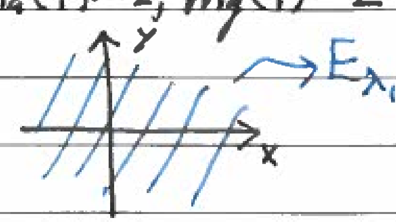
The geometric multiplicity of λ_i is

$$m_g = \dim(E_{\lambda_i}).$$

Example:

1. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \lambda_1 = 1,\ m_a(1) = 2,\ m_g(1) = 2$

$E_1 = \mathbb{R}^2.$

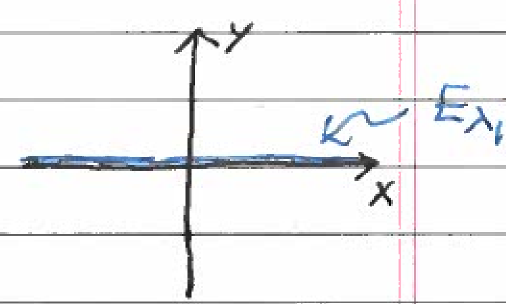


2. $A=\begin{bmatrix}1&1\\0&1\end{bmatrix}$, $\lambda_1=1$, $m_a(1)=2$, $m_g(1)=1$

$E_{\lambda_1}=\text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$:

$$A-\lambda_1 I=\begin{bmatrix}0&1\\0&0\end{bmatrix}$$

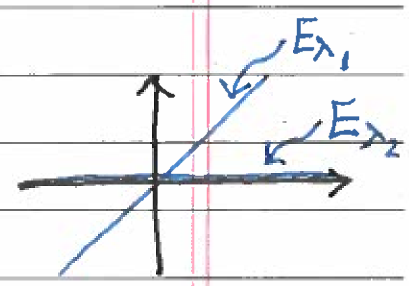
$$\Rightarrow \ker(A-\lambda_1 I)=\text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}.$$



3. $A=\begin{bmatrix}1&2\\0&3\end{bmatrix}$, $\lambda_1=1$, $\lambda_2=3$, $m_a(1)=1$, $m_a(3)=1$

$A-\lambda_1 I=\begin{bmatrix}0&2\\0&2\end{bmatrix}$, $A-\lambda_2 I=\begin{bmatrix}-2&2\\0&0\end{bmatrix}$

$\Rightarrow E_{\lambda_1}=\text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$ $\Rightarrow E_{\lambda_2}=\text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$.



Diagonalization Returned:

If there are n-linearly independent eigenvectors $\vec{v}_i$ with eigenvalues $\lambda_i \in F$ for a matrix $A \in M_{n\times n}(V)$ then if we let

$$X=[\vec{v}_1|\cdots|\vec{v}_n],\quad \Lambda=\begin{bmatrix}\lambda_1&&0\\&\ddots&\\0&&\lambda_n\end{bmatrix}$$

it follows that

$$AX=\Lambda X=X\Lambda$$

Definition of Eigenvectors

diagonal matrices Commute.

$$\Rightarrow \underbrace{A=X\Lambda X^{-1}}_{\text{Diagonalization.}}$$

Theorem - Let T be a linear transformation with eigenvalues $\{\lambda_1,\ldots,\lambda_k\}$. T is diagonalizable if and only if $m_g(\lambda_i)=\dim(E_{\lambda_i})=m_a(\lambda_i)$

Theorem - Let $T: V \to V$ be a linear transformation. If $\lambda_1, \dots, \lambda_n$ are distinct eigenvalues with corresponding eigenvectors of T then $\{\vec{v}_1, \dots, \vec{v}_n\}$ are linearly independent.

proof:

Let $P(m)$ be the logical statement that $\{\vec{v}_1, \dots, \vec{v}_n\}$ are linearly independent.

1. $P(1)$ is the statement $\{\vec{v}_1\}$ is linearly independent which is clearly true.
2. Suppose $P(m)$ is true. Therefore $\{\vec{v}_1, \dots, \vec{v}_m\}$ are linearly independent. Now, consider the equation

$$c_1\vec{v}_1 + \dots + c_{m+1}\vec{v}_{m+1} = 0 \quad (1)$$

$$\Rightarrow \lambda_1 c_1 \vec{v}_1 + \dots + \lambda_{m+1} c_{m+1} \vec{v}_{m+1} = 0. \quad (2)$$

Multiplying the first equation by λ_{m+1} and subtracting we have

$$c_1(\lambda_1 - \lambda_{m+1})\vec{v}_1 + \dots + c_m(\lambda_m - \lambda_{m+1})\vec{v}_m = 0.$$

Since $\{\vec{v}_1, \dots, \vec{v}_m\}$ are linearly independent and λ_i are distinct it follows that

$$c_1 = c_2 = \dots = c_m = 0.$$

Therefore by (1), $c_{m+1} = 0$ as well.

By items 1-2 and the principle of mathematical induction it follows that $\{\vec{v}_1, \dots, \vec{v}_K\}$ are linearly dependent.