Lecture #11: Inner Product Spaces

Definition - A complex (real) inner product on a vector space V is a mapping $\langle \cdot,\cdot \rangle : V\times V \to \mathbb{C}$ $(\mathbb{R})$ satisfying

1. $\langle c\vec{u}, \vec{v}\rangle = \bar{c}\langle \vec{u},\vec{v}\rangle$, $c\in\mathbb{C}$, $\vec{u},\vec{v}\in\mathbb{C}$.
2. $\langle \vec{u}+\vec{v},\vec{w}\rangle = \langle \vec{u},\vec{w}\rangle + \langle \vec{v},\vec{w}\rangle$, $\vec{u},\vec{v},\vec{w}\in V$.
3. $\langle \vec{u},\vec{v}\rangle = \overline{\langle \vec{v},\vec{u}\rangle}$, $\vec{u},\vec{v}\in V$.
4. $\langle \vec{u},\vec{u}\rangle = \|\vec{u}\|^2 \in\mathbb{R}$, $\langle \vec{u},\vec{v}\rangle \geq 0$ with equality $\Leftrightarrow \vec{u}=0$.

Examples -

1. Let $\vec{u},\vec{v}\in\mathbb{C}^n$, the standard inner product is defined by
$$\langle \vec{u},\vec{v}\rangle = \vec{u}\cdot\vec{v} = \vec{u}^*\vec{v} = \bar{u}_1 v_1 + \dots + \bar{u}_n v_n.$$
The operation $*$ is the conjugate transpose and for $A\in M_{m\times n}(\mathbb{C})$ is defined by
$$A^* = \begin{bmatrix} \bar{a}_{11} & \cdots & \bar{a}_{1n} \\ \vdots & & \vdots \\ \bar{a}_{m1} & \cdots & \bar{a}_{mn} \end{bmatrix}^T$$
Note, for $c\in\mathbb{C} \cong M_{1\times 1}(\mathbb{C})$, $c^* = \bar{c}$.

2. Let $f,g\in C([a,b];\mathbb{R})$
$$\langle f,g\rangle = \int_a^b f(x)g(x)\,dx$$
is a real inner product.

Definition - For $\vec{v}\in V$ with inner product $\langle\cdot,\cdot\rangle$, the norm is defined by
$$\|\vec{v}\| = \langle \vec{v},\vec{v}\rangle^{1/2}.$$

Example:

If V is a vector space with inner product $\langle\cdot,\cdot\rangle$, then for all $\vec{u},\vec{v}\in V$ and $c\in\mathbb{C}$ $c\langle\vec{u},\vec{v}\rangle=\langle\vec{u},c\vec{v}\rangle$.

proof:

$$c\langle\vec{u},\vec{v}\rangle=c\,\overline{\langle\vec{v},\vec{u}\rangle}=\overline{\bar{c}\langle\vec{v},\vec{u}\rangle}=\overline{\langle c\vec{v},\vec{u}\rangle}=\langle\vec{u},c\vec{v}\rangle.$$

$\blacksquare$

Example:

If V is a vector space with inner product $\langle\cdot,\cdot\rangle$ then for all $\vec{u},\vec{v},\vec{w},\vec{z}\in V$

$$\begin{aligned}\langle\vec{u}+\vec{v},\vec{w}+\vec{z}\rangle&=\langle\vec{u},\vec{w}+\vec{z}\rangle+\langle\vec{v},\vec{w}+\vec{z}\rangle\\&=\overline{\langle\vec{w}+\vec{z},\vec{u}\rangle}+\overline{\langle\vec{w}+\vec{z},\vec{v}\rangle}\\&=\overline{\langle\vec{w},\vec{u}\rangle}+\overline{\langle\vec{z},\vec{u}\rangle}+\overline{\langle\vec{w},\vec{v}\rangle}+\overline{\langle\vec{z},\vec{v}\rangle}\\&=\langle\vec{u},\vec{w}\rangle+\langle\vec{u},\vec{z}\rangle+\langle\vec{v},\vec{w}\rangle+\langle\vec{v},\vec{z}\rangle.\end{aligned}$$

Definition - If V is a vector space with inner product $\langle\cdot,\cdot\rangle$ then $\vec{u},\vec{v}$ are orthogonal if $\langle\vec{u},\vec{v}\rangle=0$.

Definition - A set of vectors $\{\vec{v}_1,\dots,\vec{v}_k\}$ is orthogonal with respect to an inner product $\langle\cdot,\cdot\rangle$ if $\langle\vec{v}_i,\vec{v}_j\rangle=0$ when $i\neq j$ and each $\vec{v}_i\neq 0$. In addition the set is orthonormal if $\|v_i\|=1$.

Example:

If $\{\vec{v}_1,\dots,\vec{v}_n\}$ are orthogonal then $\{\vec{v}_1,\dots,\vec{v}_n\}$ are linearly independent and hence a basis for $\operatorname{span}\{\vec{v}_1,\dots,\vec{v}_n\}$.

proof:

Suppose $c_1\vec{v}_1+\dots+c_n\vec{v}_n=0$. Therefore,

$$\begin{aligned}&\langle\vec{v}_m,c_1\vec{v}_1+\dots+c_n\vec{v}_n\rangle=0\\&\Rightarrow c_1\langle\vec{v}_m,\vec{v}_1\rangle+\dots+c_m\langle\vec{v}_m,\vec{v}_m\rangle+\dots+c_n\langle\vec{v}_m,\vec{v}_n\rangle=0\\&\Rightarrow c_m\|\vec{v}_m\|^2=0\Rightarrow c_m=0.\end{aligned}$$

$\blacksquare$

Example:

If $\{\vec{v}_1, \ldots, \vec{v}_n\} = \beta$ is an orthonormal basis then in the β basis

$$\vec{u} = \langle \vec{v}_1, \vec{u} \rangle \vec{v}_1 + \ldots + \langle \vec{v}_n, \vec{u} \rangle \vec{v}_n.$$

~~proof~~:

Let $\vec{u} = a_1\vec{v}_1 + \ldots + a_n\vec{v}_n$. Therefore,

$$\begin{aligned}\langle \vec{v}_i, \vec{u} \rangle &= \langle \vec{v}_i, a_1\vec{v}_1 + \ldots + a_n\vec{v}_n \rangle \\ &= a_1\langle \vec{v}_i, \vec{v}_1 \rangle + \ldots + a_i\langle \vec{v}_i, \vec{v}_i \rangle + \ldots + a_n\langle \vec{v}_i, \vec{v}_n \rangle \\ &= a_i.\end{aligned}$$

■