# Lecture #12: Orthogonal Compliment and Gram-Schmidt

**Definition** - Let $V$ be a vector space with inner product $\langle\cdot,\cdot\rangle$ and $S\subseteq V$. The orthogonal compliment of $S$ is defined by

$$S^{\perp}=\{\vec{v}\in V : \langle\vec{u},\vec{v}\rangle=0 \text{ for all } \vec{u}\in S\}.$$

**Example:**

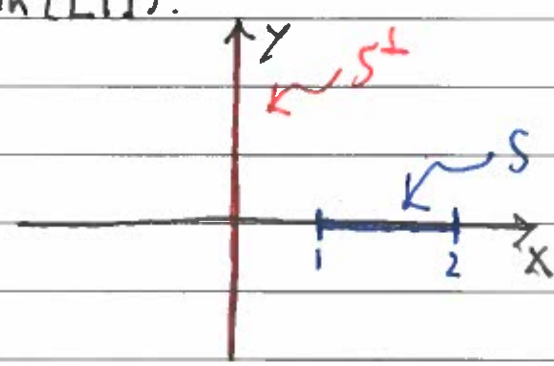
Let $V=\mathbb{R}^2$ with the standard inner product and let

$$S=\left\{\begin{bmatrix}x\\y\end{bmatrix}\in\mathbb{R}^2 : y=0 \text{ and } x\in[1,2]\right\}.$$

Note, $S$ is not a subspace since it does not contain $\vec{0}$.

Therefore,

$$\begin{aligned}S^{\perp}&=\left\{\begin{bmatrix}w\\z\end{bmatrix}\in\mathbb{R}^2 : \left\langle\begin{bmatrix}w\\z\end{bmatrix},\begin{bmatrix}x\\0\end{bmatrix}\right\rangle \text{ where } x\in[1,2]\right\}\\
&=\left\{\begin{bmatrix}w\\z\end{bmatrix}\in\mathbb{R}^2 : w\cdot x=0 \text{ where } x\in[1,2]\right\}\\
&=\left\{\begin{bmatrix}w\\z\end{bmatrix}\in\mathbb{R}^2 : w=0\right\}\\
&=\operatorname{span}\left\{\begin{bmatrix}0\\1\end{bmatrix}\right\}.\end{aligned}$$




**Theorem** - If $V$ is a vector space with inner product $\langle\cdot,\cdot\rangle$ and $S\subseteq V$, then $S^{\perp}$ is a vector space.

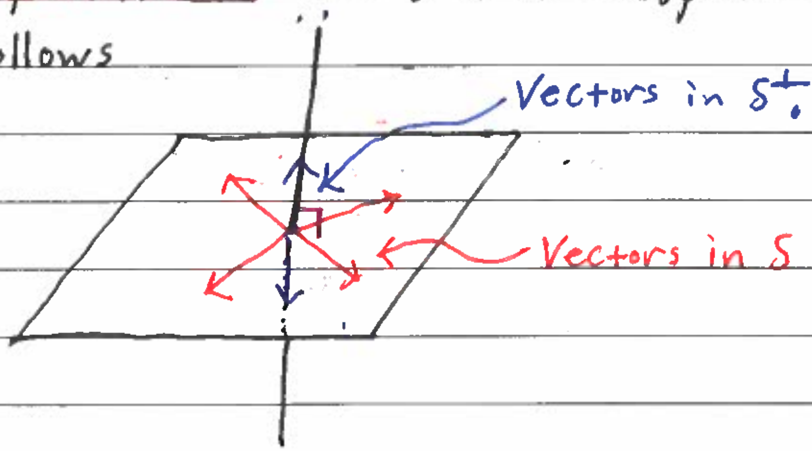
*proof:*

Let $\vec{u},\vec{v}\in S^{\perp}$ and $\lambda\in F$. Therefore, for all $\vec{w}\in S$ it follows that $\langle\vec{u},\vec{w}\rangle=0$ and $\langle\vec{v},\vec{w}\rangle=0$. Consequently,

$$\langle\vec{w},\vec{u}+\lambda\vec{v}\rangle=\langle\vec{w},\vec{u}\rangle+\lambda\langle\vec{w},\vec{v}\rangle=0$$

and thus $\vec{u}+\lambda\vec{v}\in S^{\perp}$. ∎

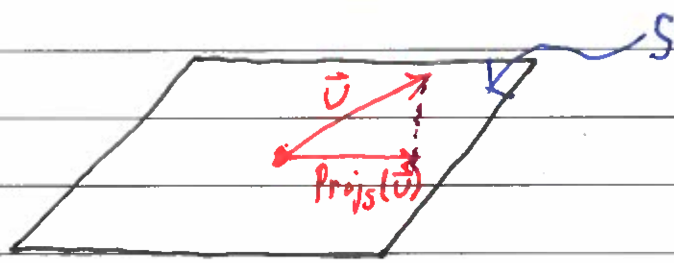
**Graphical Representation** - If $S$ is a subspace we can visualize $S^\perp$ as follows




**Definition** - If $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is an orthonormal basis and $S = \text{span}\{\vec{v}_1, \ldots, \vec{v}_n\}$ then the projection of $\vec{U} \in V$ onto $S$, denoted $\text{Proj}_S(\vec{U})$ is defined by

$$\text{Proj}_S(\vec{U}) = \langle \vec{v}_1, \vec{U} \rangle \vec{v}_1 + \ldots + \langle \vec{v}_n, \vec{U} \rangle \vec{v}_n$$

coordinates of $\vec{U}$'s shadow on $S$.



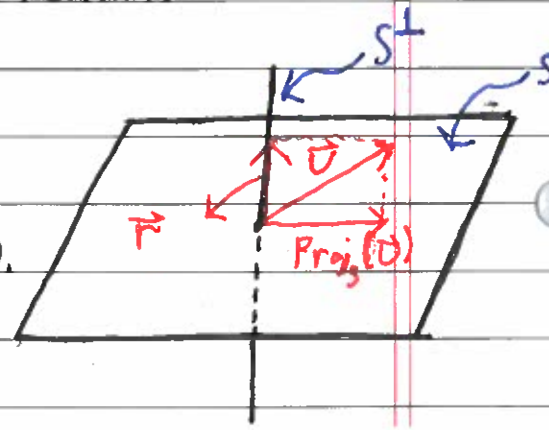

**Motivation:**

Let $\{\vec{v}_1, \ldots, \vec{v}_n\}$ be an orthonormal basis for $S$ and $\{\vec{w}_1, \ldots, \vec{w}_m\}$ an orthonormal basis for $S^\perp$. Then

$$\vec{U} = a_1\vec{v}_1 + \ldots + a_n\vec{v}_n + b_1\vec{w}_1 + \ldots + b_m\vec{w}_m.$$

$$\Rightarrow \langle \vec{v}_i, \vec{U} \rangle = \langle \vec{v}_i, a_1\vec{v}_1 + \ldots + a_n\vec{v}_n + b_1\vec{w}_1 + \ldots + b_m\vec{w}_m \rangle$$

$$= a_i \langle \vec{v}_i, \vec{v}_i \rangle = a_i.$$




That is we can write:

$$\vec{U} = \text{Proj}_S(\vec{U}) + \vec{r} = \text{Proj}_S(\vec{U}) + \text{Proj}_{S^\perp}(\vec{U}).$$

residual

Theorem - Suppose $S = \text{span}\{\vec{v}_1, \dots, \vec{v}_n\}$, where $\{\vec{v}_1, \dots, \vec{v}_n\}$ is an orthonormal set. If $\vec{u} \in V$ and $\vec{r} = \vec{u} - \text{Proj}_S(\vec{u})$ then $\vec{r} \in S^{\perp}$.

proof:

$$\begin{aligned}
\langle \vec{r}, \vec{v}_i \rangle &= \langle \vec{u} - \text{Proj}_S(\vec{u}), \vec{v}_i \rangle \\
&= \langle \vec{u} - \langle \vec{v}_1, \vec{u} \rangle \vec{v}_1 - \dots - \langle \vec{v}_n, \vec{u} \rangle \vec{v}_n, \vec{v}_i \rangle \\
&= \langle \vec{u}, \vec{v}_i \rangle - \overline{\langle \vec{v}_1, \vec{u} \rangle} \cancelto{0}{\langle \vec{v}_1, \vec{v}_i \rangle} - \dots - \overline{\langle \vec{v}_i, \vec{u} \rangle} \cancelto{1}{\langle \vec{v}_i, \vec{v}_i \rangle} \\
&\quad - \dots - \overline{\langle \vec{v}_n, \vec{u} \rangle} \cancelto{0}{\langle \vec{v}_n, \vec{v}_i \rangle} \\
&= \langle \vec{u}, \vec{v}_i \rangle - \langle \vec{v}_i, \vec{u} \rangle \\
&= \langle \vec{u}, \vec{v}_i \rangle - \langle \vec{u}, \vec{v}_i \rangle \\
&= 0.
\end{aligned}$$

## Gram-Schmidt Orthogonalization

Let $\beta = \{\vec{v}_1, \dots, \vec{v}_n\}$. Find an orthonormal set $\{\vec{u}_1, \dots, \vec{u}_n\}$ such that $\text{span}\{\vec{u}_1, \dots, \vec{u}_n\} = \text{span}\{\vec{v}_1, \dots, \vec{v}_n\}$.

1. Set $\vec{u}_1 = \vec{v}_1 / \|\vec{v}_1\|$.

2. Let $\vec{w}_2 = \vec{v}_2 - \langle \vec{u}_1, \vec{v}_2 \rangle \vec{u}_1$.

$$\begin{aligned}
&\Rightarrow \vec{w}_2 \in \text{span}\{\vec{v}_2, \vec{u}_1\} = \text{span}\{\vec{v}_2, \vec{v}_1\} \\
&\Rightarrow \langle \vec{w}_2, \vec{u}_1 \rangle = \langle \vec{v}_2 - \langle \vec{u}_1, \vec{v}_2 \rangle \vec{u}_1, \vec{u}_1 \rangle \\
&\qquad = \langle \vec{v}_2, \vec{u}_1 \rangle - \langle \vec{u}_1, \vec{v}_2 \rangle \langle \vec{u}_1, \vec{u}_1 \rangle \\
&\qquad = \langle \vec{v}_2, \vec{u}_1 \rangle - \langle \vec{v}_2, \vec{u}_1 \rangle \\
&\qquad = 0.
\end{aligned}$$

Therefore, if we let

$$\vec{u}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|}$$

we have $\langle u_2, u_1 \rangle = 0$ and

$$\text{span}\{\vec{u}_1, \vec{u}_2\} = \text{span}\{\vec{v}_1, \vec{v}_2\}.$$

3. $\vec{w}_3 = \vec{v}_3 - \langle \vec{u}_1, \vec{v}_3 \rangle \vec{u}_1 - \langle \vec{u}_2, \vec{v}_3 \rangle \vec{u}_2$

$\Rightarrow \vec{w}_3 \in \operatorname{span}\{\vec{v}_3, \vec{u}_1, \vec{u}_2\} = \operatorname{span}\{\vec{v}_3, \vec{v}_2, \vec{v}_1\}$

Set $\vec{u}_3 = \dfrac{\vec{w}_3}{\|\vec{w}_3\|}$ and continue until you are done.

Example:

Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. Find orthonormal vectors $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$

Such that $\operatorname{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3\} = \operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

1. Set $\vec{u}_1 = \dfrac{\vec{v}_1}{\|\vec{v}_1\|} = \dfrac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

2. Set $\vec{w}_2 = \vec{v}_2 - \langle \vec{u}_1, \vec{v}_2 \rangle \vec{u}_1$

$$= \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{3}{2} \cdot \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 \\ 1 \\ -3 \\ 1 \end{bmatrix}$$

$\Rightarrow \|\vec{w}_2\|^2 = \langle \vec{w}_2, \vec{w}_2 \rangle = \dfrac{1}{16} \cdot 12 = \dfrac{3}{4}$

$\Rightarrow \|\vec{w}_2\| = \sqrt{3}/2$

$\Rightarrow \|\vec{u}_2\| = \dfrac{2}{\sqrt{3}} \cdot \dfrac{1}{4} \begin{bmatrix} 1 \\ 1 \\ -3 \\ 1 \end{bmatrix} = \dfrac{1}{2\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -3 \\ 1 \end{bmatrix}$

3. $\vec{w}_3 = \vec{v}_3 - \langle \vec{u}_1, \vec{v}_3 \rangle \vec{u}_1 - \langle \vec{u}_2, \vec{v}_3 \rangle \vec{u}_2$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \cdot 2 \cdot \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2\sqrt{3}} \cdot 2 \cdot \frac{1}{2\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{6} \begin{bmatrix} 1 \\ 1 \\ -3 \\ 1 \end{bmatrix}$$

$$= \frac{1}{6}\left( \begin{bmatrix} 6 \\ 0 \\ 0 \\ 6 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ -3 \\ 1 \end{bmatrix} \right)$$

$$\Rightarrow \vec{w}_3 = \frac{1}{6}\begin{bmatrix} 2 \\ -4 \\ 0 \\ 2 \end{bmatrix}$$

$$\Rightarrow \langle \vec{w}_3, \vec{w}_3 \rangle = \frac{1}{36} 20 = \frac{5}{9}.$$

Consequently, $\|\vec{w}_3\| = \sqrt{5}/3$

$$\vec{v}_3 = \frac{\vec{w}_3}{\|\vec{w}_3\|} = \frac{3}{\sqrt{5}} \cdot \frac{1}{6}\begin{bmatrix} 2 \\ -4 \\ 0 \\ 2 \end{bmatrix} = \frac{1}{\sqrt{5}\cdot 2}\begin{bmatrix} 2 \\ -4 \\ 0 \\ 2 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}.$$

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Example:

Let $\beta = \{1, x, x^2\}$ be a basis for $P_2(\mathbb{R})$ with the inner product

$$\langle p(x), q(x) \rangle = \int_{-1}^{1} p(x)q(x)\,dx.$$

Find an orthonormal basis for $P_2(\mathbb{R})$.

1. Let $\vec{v}_1 = 1/\|1\|$. Now, $\|1\|^2 = \langle 1, 1 \rangle = \int_{-1}^{1} 1\,dx = 2$. Therefore,
$$\vec{v}_1 = 1/\sqrt{2}.$$

2. Let $\vec{w}_2 = x - \langle \vec{v}_1, x \rangle \vec{v}_1$. Now,
$$\langle \vec{v}_1, x \rangle = \int_{-1}^{1} x/\sqrt{2}\,dx = x^2/2\sqrt{2}\Big|_{-1}^{1} = 0.$$
Therefore,
$$\vec{w}_2 = x.$$
Now,
$$\|\vec{w}_2\|^2 = \langle \vec{w}_2, \vec{w}_2 \rangle = \int_{-1}^{1} x^2\,dx = 2\int_0^1 x^2\,dx = 2/3.$$
Therefore,
$$\vec{v}_2 = \sqrt{3/2}\,x$$

3. Let $\vec{w}_3 = x^2 - \langle \vec{v}_1, x^2 \rangle \vec{v}_1 - \langle \vec{v}_2, x^2 \rangle \vec{v}_2$. Now,
$$\langle v_1, x^2 \rangle = 1/\sqrt{2}\int_{-1}^{1} x^2\,dx = 2/\sqrt{2}\int_0^1 x^2\,dx = 2/\sqrt{2}\cdot 3 = \sqrt{2}/3$$
$$\langle v_2, x^2 \rangle = \sqrt{3/2}\int_{-1}^{1} x^3\,dx = 0.$$

Therefore,

$$\vec{w}_3 = x^2 - \sqrt{2}/3 \cdot 1/\sqrt{2} = x^2 - 1/3$$

Consequently, $\|\vec{w}_3\|^2 = \langle \vec{w}_3, \vec{w}_3 \rangle = \int_{-1}^{1} (x^2 - 1/3)^2 dx$

$$= 2\int_0^1 (x^2 - 1/3)^2 dx$$

$$= 2\int_0^1 (x^4 - 2/3 x^2 + 1/9) dx$$

$$= 2(1/5 - 2/9 + 1/9)$$

$$= 2\left(\frac{9 - 10 + 5}{45}\right)$$

$$= 8/45$$

Therefore,

$$\|w_3\| = 2\sqrt{2}/3\sqrt{5}.$$

Consequently,

$$\vec{u}_3 = \frac{\vec{w}_3}{\|\vec{w}_3\|} = \frac{3\sqrt{5}}{2\sqrt{2}} \cdot (x^2 - 1/3) = \frac{\sqrt{10}}{4}(3x^2 - 1).$$