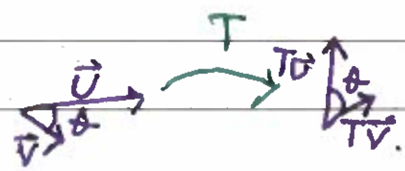
Lecture #13: Unitary Matrices

Definition - A linear transformation $T: V \to V$ is unitary or an isometry with respect to an inner product $\langle \cdot, \cdot \rangle$ if for all $\vec{u}, \vec{v} \in V$

$$\langle T\vec{u}, T\vec{v} \rangle = \langle \vec{u}, \vec{v} \rangle$$

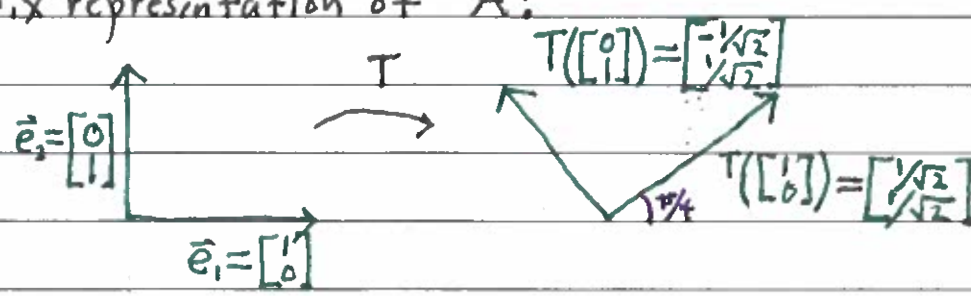
Motivation -

1. For a real inner product
$$\langle T\vec{u}, T\vec{v} \rangle = \langle \vec{u}, \vec{v} \rangle = \|\vec{u}\| \cdot \|\vec{v}\| \cos\theta \Rightarrow \text{preserves angles.}$$
2. For any inner product
$$\|T\vec{u}\|^2 = \langle T\vec{u}, T\vec{u} \rangle = \langle \vec{u}, \vec{u} \rangle = \|\vec{u}\|^2 \Rightarrow \text{preserves lengthes}$$



Example -

Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a rotation by $\pi/4$ radians. What is the matrix representation of A?



Therefore,
$$A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Theorem - A matrix $A \in M_{n\times n}(\mathbb{C})$ is unitary with respect to the complex inner product $\langle \cdot , \cdot \rangle$ if and only if the columns of A are orthonormal.

proof:

The standard basis $\{\vec{e}_1, \dots, \vec{e}_n\}$ is orthormal. Since

$$A = [A(\vec{e}_1) | \cdots | A(\vec{e}_n)]$$

and $\langle A(\vec{e}_i), A(\vec{e}_j) \rangle = \langle \vec{e}_i, \vec{e}_j \rangle = \begin{cases} 1, & \text{if } i=j \\ 0, & \text{if } i \neq j \end{cases}$.

Example:

$$A = \begin{bmatrix} (1+i)/2 & (1-i)/2 \\ (1+i)/2 & (-1+i)/2 \end{bmatrix}$$

$$\vec{q}_1 = \frac{1}{2}\begin{bmatrix} 1+i \\ 1+i \end{bmatrix}, \quad \vec{q}_2 = \frac{1}{2}\begin{bmatrix} 1-i \\ -1+i \end{bmatrix}$$

$$\langle \vec{q}_1, \vec{q}_2 \rangle = \vec{q}_1^{\,*}\vec{q}_2 = \frac{1}{4}\begin{bmatrix} 1-i & 1-i \end{bmatrix}\begin{bmatrix} 1-i \\ -1+i \end{bmatrix} = \frac{1}{4}\left((1-i)^2 - (1-i)^2\right) = 0.$$

$$\langle \vec{q}_1, \vec{q}_1 \rangle = \vec{q}_1^{\,*}\vec{q}_1 = \frac{1}{4}\begin{bmatrix} 1-i & 1-i \end{bmatrix}\begin{bmatrix} 1+i \\ 1+i \end{bmatrix} = \frac{1}{4}(2+2) = 1.$$

$$\langle \vec{q}_2, \vec{q}_2 \rangle = \vec{q}_2^{\,*}\vec{q}_2 = \frac{1}{4}\begin{bmatrix} 1+i & -1-i \end{bmatrix}\begin{bmatrix} 1-i \\ -1+i \end{bmatrix} = \frac{1}{4}(2+2) = 1.$$

Theorem - A is unitary if and only if $A^* = A^{-1}$.

proof:

$A = [\vec{q}_1 | \cdots | \vec{q}_n]$ is unitary if and only if $\{\vec{q}_1, \dots, \vec{q}_n\}$ are orthonormal.

Therefore, A is unitary if and only if

$$A^*A = \begin{bmatrix} \vec{q}_1^{\,*} \\ \vdots \\ \vec{q}_n^{\,*} \end{bmatrix}[\vec{q}_1 | \cdots | \vec{q}_n] = \begin{bmatrix} \vec{q}_1^{\,*}\vec{q}_1 & \cdots & \vec{q}_1^{\,*}\vec{q}_n \\ \vdots & & \vdots \\ \vec{q}_n^{\,*}\vec{q}_1 & \cdots & \vec{q}_n^{\,*}\vec{q}_n \end{bmatrix} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 1 \end{bmatrix} = I.$$

$\blacksquare$

Theorem - The eigenvalues of a unitary matrix satisfy $|\lambda|=1$.

~~proof~~:

Let $\vec{v}$ be an eigenvector with eigenvalue λ. Therefore,

$$\langle A\vec{v}, A\vec{v}\rangle = \langle \vec{v}, \vec{v}\rangle$$

$$\langle \lambda\vec{v}, \lambda\vec{v}\rangle = \lambda\bar{\lambda}\langle \vec{v}, \vec{v}\rangle$$

$$\Rightarrow \|\vec{v}\|^2 = |\lambda|^2 \|\vec{v}\|^2$$

$$\Rightarrow |\lambda|^2 = 1$$

$$\Rightarrow |\lambda| = 1.$$

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Theorem - Eigenvectors corresponding to distinct eigenvalues are orthogonal.

~~proof~~:

Let $\vec{v}_1, \vec{v}_2$ be eigenvectors with corresponding eigenvalues λ_1, λ_2. Therefore

$$\langle A\vec{v}_1, A_2\vec{v}_2\rangle = \langle \vec{v}_1, \vec{v}_2\rangle$$

$$\langle \lambda_1\vec{v}_1, \lambda_2\vec{v}_2\rangle = \lambda_1\bar{\lambda}_2\langle \vec{v}_1, \vec{v}_2\rangle$$

$$\Rightarrow \lambda_1\bar{\lambda}_2\langle \vec{v}_1, \vec{v}_2\rangle = \langle \vec{v}_1, \vec{v}_2\rangle$$

Since $\lambda_1, \lambda_2 \neq 0$ it follows that $\langle \vec{v}_1, \vec{v}_2\rangle = 0$.

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Spectral Theorem - If $A \in M_{n \times n}(\mathbb{C})$ is a unitary matrix with distinct eigenvalues $\lambda_1, \dots, \lambda_n$ then there exists a unitary matrix U such that

$$A = U \Delta U^*$$

where

$$\Delta = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}.$$

~~proof~~:

Let $\vec{v}_1, \dots, \vec{v}_n$ be the eigenvectors satisfying $\|\vec{v}_i\| = 1$. Let $U = [\vec{v}_1 | \cdots | \vec{v}_n]$, which is unitary since the columns are orthonormal. Therefore,

$$A \cdot U = U \Delta$$

$$\Rightarrow A = U \Delta U^*$$

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