Lecture #14: Rank 1 Matrices

Theorem - A matrix $A \in M_{n\times n}(\mathbb{C})$ is rank 1 if and only if there exists vectors $\vec{u}, \vec{v} \in \mathbb{C}^n$ such that

$$A = \vec{u}\vec{v}^*.$$

proof:

A is rank 1 if only if there exists $v_1, \dots, v_n \in \mathbb{C}$ and $\vec{u} \in \mathbb{C}^n$ such that

$$A = [\bar{v}_1 \vec{u} | \cdots | \bar{v}_n \vec{u}]$$

$$= \begin{bmatrix} \bar{v}_1 u_1 & \cdots & \bar{v}_n u_1 \\ \vdots & & \vdots \\ \bar{v}_1 u_n & \cdots & \bar{v}_n u_n \end{bmatrix}$$

$$= \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} [\bar{v}_1 \cdots \bar{v}_n]$$

$$= \vec{u} \cdot \vec{v}^*,$$

where $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$.

Theorem - If A is a rank 1 matrix such that $A = \vec{u}\vec{v}^*$ then

$$\underset{\substack{n\times n \\ \text{Storage}}}{A}\underset{\substack{n\times 1 \\ \text{Storage}}}{\vec{w}} = \vec{u}\underbrace{\vec{v}^*\vec{w}}_{\text{Scalar}} = \langle \underset{\substack{n\times 1 \\ \text{Storage}}}{\vec{v}}, \underset{\substack{n\times 1 \\ \text{Storage}}}{\vec{w}} \rangle \underset{\substack{n\times 1 \\ \text{Storage}}}{\vec{u}}$$

n^2 multiplications $+ n$ additions

$2n$ multiplications $+ n$ additions.

Example:

1. $\mathrm{Proj}_{\vec{U}}(\vec{V}) = \langle \vec{U}, \vec{V} \rangle \vec{U} = (\vec{U}^* \vec{V}) \vec{U} = \vec{U}\vec{U}^* \vec{V}$

The matrix representation of $\mathrm{Proj}_{\vec{U}}(\cdot)$ is $\vec{U}\vec{U}^*$.

2. Let $S = \mathrm{span}\{\vec{U}_1, \dots, \vec{U}_n\}$, where $\vec{U}_1, \dots, \vec{U}_n$ are orthonormal, then

$$\begin{aligned}\mathrm{Proj}_S(\vec{V}) &= \langle \vec{U}_1, \vec{V} \rangle \vec{U}_1 + \dots + \langle \vec{U}_n, \vec{V} \rangle \vec{U}_n \\ &= (\vec{U}_1^* \vec{V}) \vec{U}_1 + \dots + (\vec{U}_n^* \vec{V}) \vec{U}_n \\ &= (\vec{U}_1 \vec{U}_1^* + \dots + \vec{U}_n \vec{U}_n^*) \vec{V}\end{aligned}$$

The matrix representation of $\mathrm{Proj}_S(\cdot)$ is $\vec{U}_1\vec{U}_1^* + \dots + \vec{U}_n\vec{U}_n^*$.

3. $$\begin{aligned}\mathrm{Proj}_{S^\perp}(\vec{V}) &= \vec{V} - (\langle \vec{U}_1, \vec{V} \rangle \vec{U}_1 + \dots + \langle \vec{U}_n, \vec{V} \rangle \vec{U}_n) \\ &= (I - (\vec{U}_1 \vec{U}_1^* + \dots + \vec{U}_n \vec{U}_n^*)) \vec{V}\end{aligned}$$

The matrix representation of $\mathrm{Proj}_{S^\perp}(\cdot)$ is $I - (\vec{U}_1\vec{U}_1^* + \dots + \vec{U}_n\vec{U}_n^*)$.