Lecture 2: Complex Numbers

Define a new number i by

$$\sqrt{-1} = i. \Leftrightarrow i^2 = -1 \Leftrightarrow i \cdot i = -1.$$

The number i called the imaginary unit.

Example:

1. $i^2 = -1$
2. $i^3 = i \cdot i^2 = -i$
3. $i^4 = i^2 \cdot i^2 = -1 \cdot (-1) = 1.$
4. $i^{46} = (i^4)^{11} \cdot i^2 = 1 \cdot (-1) = -1.$
5. If $n \in \mathbb{N}$, then

$$i^n = \begin{cases} 1 & \text{if remainder}(n/4) = 0 \\ i & \text{if remainder}(n/4) = 1 \\ -1 & \text{if remainder}(n/4) = 2 \\ -i & \text{if remainder}(n/4) = 3 \end{cases}$$

Definition: A complex number is a number of the form

$$z = a + bi, \quad \text{(standard form)}$$

where $a, b \in \mathbb{R}$.

(i) a is called the real part, denoted $\mathrm{Re}(a+bi) = a$.

(ii) b is called the imaginary part, denoted $\mathrm{Im}(a+bi) = b$.

The standard notation for the set of complex numbers is $\mathbb{C}$.

Rules needed for linear algebra:

1. Addition: $(a+bi) + (c+di) = a + c + (b+d)i.$
2. Subtraction: $(a+bi) - (c+di) = a - c + (b-d)i.$
3. Multiplication: $(a+bi)(c+di) = ac + bci + adi - bd$

$$= ac - bd + i(bc + ad)$$

4. Division: $\frac{a+bi}{c+di} = \frac{a+bi}{c+di} \cdot \frac{c-di}{c-di} = \frac{ac+bd+i(bc-ad)}{c^2+d^2}$

$$\Rightarrow \frac{a+bi}{c+di} = \frac{ac+bd}{c^2+d^2} + \frac{(bc-ad)}{c^2+d^2}\, i$$

Example:

Solve the system of equations

$$(1+i)x + iy = 1$$
$$ix + y = i$$

$$\Rightarrow \left[\begin{array}{cc|c} 1+i & i & 1 \\ i & 1 & i \end{array}\right] \begin{array}{l} /1+i \\ /i \end{array} = \left[\begin{array}{cc|c} 1 & i/1+i & 1/1+i \\ 1 & 1/i & 1 \end{array}\right] = \left[\begin{array}{cc|c} 1 & (1+i)/2 & (1-i)/2 \\ 1 & -i & 1 \end{array}\right] \begin{array}{l} \\ -R1 \end{array}$$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & \frac{1}{2}+i/2 & \frac{1}{2}-\frac{i}{2} \\ 0 & -\frac{1}{2}-\frac{3}{2}i & \frac{1}{2}+\frac{i}{2} \end{array}\right] \times\left(-\frac{1}{2}+3/2\,i\right) = \left[\begin{array}{cc|c} 1 & \frac{1}{2}+i/2 & 1/2-i/2 \\ 0 & 5/2 & -1/2+i/2 \end{array}\right] \times\frac{2}{5} = \left[\begin{array}{cc|c} 1 & \frac{1}{2}+\frac{i}{2} & 1/2-i/2 \\ 0 & 1 & -2/5+i/5 \end{array}\right]$$

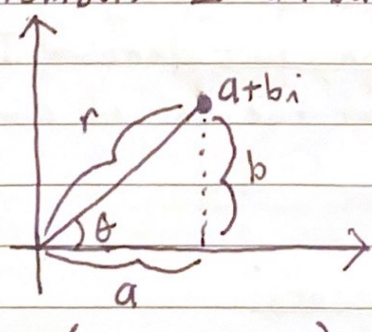
$$\Rightarrow \left[\begin{array}{cc|c} 1 & \frac{1}{2}+\frac{i}{2} & \frac{1}{2}-\frac{i}{2} \\ 0 & 1 & -\frac{2}{5}+\frac{i}{5} \end{array}\right] - \left(\frac{1}{2}+i/2\right)R2 = \left[\begin{array}{cc|c} 1 & 0 & 4/5-2/5\,i \\ 0 & 1 & -2/5+1/5\,i \end{array}\right]$$

Therefore,

$$x = 4/5 - 2/5\,i$$
$$y = -2/5 + 1/5\,i$$

Geometry:

Complex numbers $z = a+bi$ can be represented geometrically.



$z = r(\cos\theta + i\sin\theta)$. (Polar Form)

$r^2 = a^2+b^2 = (a+ib)(a-ib)$

$\bar{z} = a - ib$ (Complex Conjugate)

$\Rightarrow r = |z| = \sqrt{z\bar{z}}$ (Modulus)

Exponential Form:

$$\cos\theta = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots$$

$$e^{i\theta} = 1 + (i\theta) + \frac{(i\theta)^2}{2} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4} + \dots$$

$$= 1 + i\theta - \frac{\theta^2}{2} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \dots$$

$$= \left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} + \dots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots\right)$$

$$\Rightarrow \boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

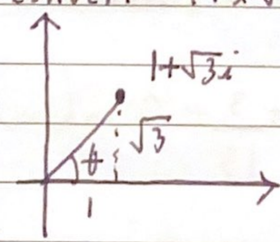
Therefore,

$z = |z|(\cos\theta + i\sin\theta) = |z|e^{i\theta}$ (Exponential Form)

Example:

Find $(1+i\sqrt{3})^8$.

First, we convert $1+i\sqrt{3}$ to exponential/polar form.



$$\theta = \tan^{-1}(\sqrt{3}) = \pi/3, \quad |z| = 2.$$

Therefore,

$$(1+i\sqrt{3})^8 = (2e^{i\pi/3})^8 = 256e^{8i\pi/3}$$

$$\Rightarrow (1+i\sqrt{3})^8 = 256(\cos(8\pi/3) + i\sin(8\pi/3))$$

$$= 256(-1/2 + i\sqrt{3}/2) = -128 + 128\cdot\sqrt{3}i.$$

Example:

Prove the following

$$\cos(\theta+\phi)=\cos(\theta)\cos(\phi)-\sin(\theta)\sin(\phi)$$
$$\sin(\theta+\phi)=\sin(\theta)\cos(\phi)+\cos(\theta)\sin(\phi)$$

~~proof~~:

$$\begin{aligned}
\cos(\theta+\phi)+i\sin(\theta+\phi) &= e^{i(\theta+\phi)} \\
&= e^{i\theta}e^{i\phi} \\
&= (\cos(\theta)+i\sin(\theta))(\cos(\phi)+i\sin(\phi)) \\
&= \cos(\theta)\cos(\phi)-\sin\theta\sin(\phi)+i(\sin(\theta)\cos(\phi)+\cos(\theta)\sin(\phi))
\end{aligned}$$

$$\Rightarrow \cos(\theta+\phi)=\cos(\theta)\cos(\phi)-\sin(\theta)\sin(\phi)$$
$$\sin(\theta+\phi)=\sin(\theta)\cos(\phi)+\cos(\theta)\sin(\phi).$$

Example:

Compute $\int e^{2x}\cos(3x)\,dx$.

We have that

$$\begin{aligned}
\int e^{2x}\cos(3x)\,dx &= \int e^{2x}\,\mathrm{Re}(e^{3ix})\,dx \\
&= \mathrm{Re}\left(\int e^{(2+3i)x}\,dx\right) \\
&= \mathrm{Re}\left(\frac{1}{2+3i}e^{(2+3i)x}\right)+C \\
&= \mathrm{Re}\left(\frac{2-3i}{13}e^{2x}(\cos(3x)+i\sin(3x))\right)+C \\
&= \frac{2}{13}e^{2x}\cos(3x)+\frac{3}{13}e^{2x}\sin(3x)+C
\end{aligned}$$

Matrix Representation

Complex number $z = a+bi$ corresponds to the matrix:

$$Z_1 = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

"~~proof~~"

Let $z = a+bi$, $w = c+di$, $Z_1 = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, $W = \begin{bmatrix} c & d \\ -d & c \end{bmatrix}$.

Therefore

$$\overbrace{a+bi+c+di}^{z+w} = a+c+i(b+d)$$

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} + \begin{bmatrix} c & d \\ -d & c \end{bmatrix} = \underbrace{\begin{bmatrix} a+c & b+d \\ -(b+d) & a+c \end{bmatrix}}_{Z_1 + W}$$

$$\overbrace{(a+ib)(c+id)}^{z \cdot w} = ac-bd+i(ad+bc)$$

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} c & d \\ -d & c \end{bmatrix} = \underbrace{\begin{bmatrix} ac-bd & ad+bc \\ -bc-ad & -bd+ac \end{bmatrix}}_{Z_1 \cdot W}$$

*The notions of subtraction, division, as well as commutativity will be addressed in homwork.