Lecture 4: Vector Spaces and Subspaces

Definition: A set V is a vector space over a field F if there are operations of addition and scalar multiplication on V such that the following properties hold.

For all $\vec{u}, \vec{v}, \vec{w} \in V$, $a, b \in F$

1. $\vec{u}+\vec{v} \in V$
2. $\vec{u}+\vec{v}=\vec{v}+\vec{u}$
3. $\vec{u}+(\vec{v}+\vec{w})=(\vec{u}+\vec{v})+\vec{w}$
4. There exists $\vec{0} \in V$ such that $\vec{0}+\vec{v}=\vec{v}$, and $\vec{0}$ is unique.
5. There is $(-\vec{u}) \in V$ such that $\vec{u}+(-\vec{u})=\vec{0}$, and $(-\vec{u})$ is unique
6. $a\vec{u} \in V$
7. $a(\vec{u}+\vec{v})=a\vec{u}+a\vec{v}$
8. $(a+b)\vec{u}=a\vec{u}+b\vec{u}$
9. $a(b\vec{u})=ab\vec{u}$
10. $1\vec{u}=\vec{u}$

Examples:

1. F^n = vectors of elements in a field F:

$$\vec{v}=\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix},$$

is a vector space over F, i.e.

$$V=\mathbb{R}^n, \mathbb{C}^n, \mathbb{Z}_p^n, \mathbb{R}^\infty, \mathbb{C}^\infty, \mathbb{Z}_p^\infty$$

sequences

2. $M_{m\times n}(F)$ = $m\times n$ matrices with entries in F is a vector space over F, i.e.

$$M_{m\times n}(\mathbb{R}), M_{m\times n}(\mathbb{C}), M_{m\times n}(\mathbb{Z}_p)$$

3. $P_n(F)$ polynomials of degree $\leq n$ with coefficients in F, i.e.

$$P_n(\mathbb{R}), P_n(\mathbb{C}), P_n(\mathbb{Z}_p), P_\infty(\mathbb{R}), P_\infty(\mathbb{C}), P_\infty(\mathbb{Z}_p)$$

are vector fields over F.

4. $\mathcal{C}(\mathbb{R})$ continuous functions from $\mathbb{R}$ to $\mathbb{R}$ is a vector space over $\mathbb{R}$.
5. $\mathcal{C}^p(\mathbb{R})$ differentiable functions whose p-th derivative is continuous are a vector space over $\mathbb{R}$.

Theorem - If V is a vector space over a field F, then for all $\vec{u} \in V$,

$$0 \cdot \vec{u} = \vec{0}$$

proof:

$$0 \cdot \vec{u} = (1+(-1))\vec{u}$$
$$= \vec{u} + (-1)\vec{u}$$
$$\Rightarrow 0 \cdot \vec{u} + \vec{u} = \vec{u}$$

Consequently, $0 \cdot \vec{u} = \vec{0}$

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Theorem - If V is a vector space over a field F, then for all $\vec{u} \in V$

$$-\vec{u} = (-1) \cdot \vec{u}$$

proof:

$$(1+(-1))\vec{u} = 0$$
$$\Rightarrow \vec{u} + (-1)\vec{u} = 0$$
$$\Rightarrow (-1)\vec{u} = -\vec{u}.$$

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Theorem - If V is a vector space over a field F then for all $c \in F$

$$c \cdot \vec{0} = \vec{0}$$

proof:

$$c \cdot \vec{0} = c(\vec{0} + \vec{0}) = c \cdot \vec{0} + c \cdot \vec{0}$$
$$\Rightarrow \vec{0} = c \cdot \vec{0}$$

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Definition - A subspace of a vector space V over F is a subset $U \subseteq V$ that is also a vector space over V with the same operations of vector addition and scalar multiplication.

Theorem - Let V be a vector space over F. U is a subspace of V if and only if $c\vec{u}+\vec{v} \in U$ whenever $\vec{u}, \vec{v} \in U$ and $c \in F$.

proof

Properties 2, 3, 7, 8, 9, 10 are satisfied by construction.

(i) Property 1 is satisfied by setting $c=1$.

(ii) Setting $c=-1$ and $\vec{u}=\vec{v}$ yields $\vec{0} \in U$. (Property 4)

(iii) Setting $c=-1$ and $\vec{v}=\vec{0}$ yields $-\vec{u} \in U$ (Property 5)

(iv) Setting $\vec{v}=\vec{0}$ yields $c\vec{u} \in V$ (Property 6)

$\blacksquare$

Examples:

1. $W=\{\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : y=x\}$ is a subspace of $\mathbb{R}^2$

proof:

Let $\vec{w}_1, \vec{w}_2 \in W$ and $a \in \mathbb{R}$. Therefore, there exists x_1, x_2 such that $\vec{w}_1=\begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$, $\vec{w}_2=\begin{bmatrix} x_2 \\ x_2 \end{bmatrix}$. Consequently

$$\vec{w}_1+a\vec{w}_2=\begin{bmatrix} x_1 \\ x_1 \end{bmatrix}+a\begin{bmatrix} x_2 \\ x_2 \end{bmatrix}=\begin{bmatrix} x_1+ax_2 \\ x_1+ax_2 \end{bmatrix} \in W.$$

Therefore, W is a subspace.

2. $W=\{2\times 2$ symmetric matrices$\}$ is a subspace of $M_{2\times 2}(\mathbb{R})$.

proof:

Let $M_1, M_2 \in W$ and $a \in \mathbb{R}$. Therefore, there exists a_1, b_1, c_1 and a_2, b_2, c_2 such that

$$M_1=\begin{bmatrix} a_1 & b_1 \\ b_1 & c_1 \end{bmatrix}, \quad M_2=\begin{bmatrix} a_2 & b_2 \\ b_2 & c_2 \end{bmatrix}.$$

Consequently,

$$M_1 + aM_2 = \begin{bmatrix} a_1 & b_1 \\ b_1 & c_1 \end{bmatrix} + a\begin{bmatrix} a_2 & b_2 \\ b_2 & c_2 \end{bmatrix} = \begin{bmatrix} a_1+aa_2 & b_1+ab_2 \\ b_1+ab_2 & c_1+ac_2 \end{bmatrix} \in W.$$

Therefore, W is a subspace.

3. $W = \{$polynomials $p(x)$ satisfying $p(1)=1\}$ is not a subspace of $P_\infty(\mathbb{R})$ since $p(x)=0 \notin W$.

4. $W = \{$continuous functions satisfying $f(0)=f(2\pi n)$, for all $n \in \mathbb{N}\}$ is a subspace of $\mathcal{C}(\mathbb{R})$.

~~proof~~:

Let $f, g \in W$. Therefore all $n \in \mathbb{N}$

$$f(0) = f(2\pi n) \text{ and } g(0) = g(2\pi n).$$

For $a \in \mathbb{R}$, let $h(x) = f(x) + ag(x)$. Consequently,

$$h(0) = f(0) + ag(0) = f(2\pi n) + ag(2\pi n) = h(2\pi n).$$

Consequently, $h \in W$ and thus W is a subspace.

5. If $A \in M_{n\times n}(\mathbb{C})$ then

$$\mathrm{Ker}(A) = \mathrm{nul}(A) = \{\vec{v} \in \mathbb{C}^n : A\vec{v} = \vec{0}\}$$

is a subspace of $\mathbb{C}^n$.

~~proof~~:

Let $\vec{u}, \vec{v} \in \mathrm{Ker}(A)$. Therefore $A\vec{u} = 0$ and $A\vec{v} = \vec{0}$. Consequently, if $\vec{w} = \vec{u} + a\vec{v}$ for some $a \in \mathbb{C}$ it follows that

$$A\vec{w} = A(\vec{u} + a\vec{v}) = A\vec{u} + aA\vec{v} = \vec{0} + a\cdot\vec{0} = \vec{0}.$$

Therefore, W is a subspace of $\mathbb{C}^n$.

6. If $A \in M_{n\times n}(\mathbb{C})$ then

$$\mathrm{im}(A) = \mathrm{col}(A) = \{\vec{w} \in \mathbb{C}^n : \text{there exists } \vec{x} \text{ such that } A\vec{x} = \vec{w}\}.$$

is a subspace of $\mathbb{C}^n$.

~~proof~~:

Let $\vec{u}, \vec{v} \in \mathrm{im}(A)$. Therefore, there exists $\vec{x}, \vec{y} \in \mathbb{C}^n$ such that $A\vec{x} = \vec{u}$, $A\vec{y} = \vec{v}$. Consequently, if $\vec{w} = \vec{u} + a\vec{v}$ for some $a \in \mathbb{C}$ it follows that

$$\vec{w} = \vec{u} + a\vec{v} = A\vec{x} + aA\vec{y} = A(\vec{x} + a\vec{y}) \in \mathrm{im}(A).$$