Lecture #5: Span, Idependence, Basis and Dimension

Definition - A vector $\vec{v} \in V$ is a <u>linear combination</u> of $\vec{v}_1, \ldots, \vec{v}_n \in V$ if it can be written in the form

$$\vec{w} = k_1\vec{v}_1 + k_2\vec{v}_2 + \ldots + k_n\vec{v}_n$$

where $k_1, \ldots, k_n \in F$.

Example:

In $C(\mathbb{R})$, is $\sin(x)$ a linear combination of e^x, x^2?

Assume for contridiction that

$$\sin(x) = k_1 e^x + k_2 x^2$$

Setting $x=0$ we have

$$0 = k_1$$

Setting $x = \pi$ we have

$$0 = k_2\pi^2 \Rightarrow k_2 = 0.$$

Therefore, we have the contradiction

$$\sin(x) = 0.$$

Definition - If V is a vector space and $\vec{v}_1, \ldots, \vec{v}_n$ are vectors in V, then the set of all linear combinations of $\vec{v}_1, \ldots, \vec{v}_n$ is a subspace of V called <u>$\text{span}\{\vec{v}_1, \ldots, \vec{v}_n\}$</u>.

Example:

In $\mathbb{R}^3$, prove that $\text{span}\left\{\begin{bmatrix}3\\1\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}\right\} = \left\{\begin{bmatrix}x\\y\\z\end{bmatrix} \in \mathbb{R}^3 : z=0\right\}$.

proof:

1. Let $\begin{bmatrix}a\\b\\c\end{bmatrix} \in \text{span}\left\{\begin{bmatrix}3\\1\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}\right\}$. Therefore, there exists $c_1, c_2 \in \mathbb{R}$ such that

$$\begin{bmatrix}a\\b\\c\end{bmatrix} = c_1\begin{bmatrix}3\\1\\0\end{bmatrix} + c_2\begin{bmatrix}2\\1\\0\end{bmatrix}$$

$$\Rightarrow c = 0$$

$$\Rightarrow \begin{bmatrix}a\\b\\c\end{bmatrix} \in \left\{\begin{bmatrix}x\\y\\z\end{bmatrix} \in \mathbb{R}^3 : z=0\right\}.$$

2. Now, let $\begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \in \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : z = 0 \right\}$. To determine if $\begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \in \text{span}\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$, we set up the system

$$\begin{bmatrix} a \\ b \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & 2 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{cc|c} 3 & 2 & a \\ 1 & 1 & b \end{array}\right] \xrightarrow{-3R2} \left[\begin{array}{cc|c} 0 & -1 & a-3b \\ 1 & 1 & b \end{array}\right] {+R1} \xrightarrow{x-1} \left[\begin{array}{cc|c} 1 & 0 & a-2b \\ 0 & 1 & -a+3b \end{array}\right]$$

Since this has a solution it follows that $\begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \in \text{span}\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$.

By items 1 and 2, $\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : z = 0 \right\} = \text{span}\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$.

<u>Definition</u>: Let $\vec{v}_1, \ldots, \vec{v}_n \in V$. The set $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is linearly independent if

$$a_1 \vec{v}_1 + \cdots + a_n \vec{v}_n = 0 \Rightarrow a_1 = \ldots = a_n = 0.$$

<u>Remark</u>: If one of the $a_i \neq 0$, say a_1, we have

$$\vec{v}_1 = -\frac{a_2}{a_1}\vec{v}_2 - \frac{a_3}{a_1}\vec{v}_3 - \ldots - \frac{a_n}{a_1}\vec{v}_n,$$

i.e., $\vec{v}_1$ depends on $\vec{v}_2, \ldots, \vec{v}_n$.

<u>Example</u>:

Suppose $\vec{u}, \vec{v}, \vec{w}, \vec{x} \in V$. Show that $\{\vec{u}-\vec{v}, \vec{v}-\vec{w}, \vec{w}-\vec{x}, \vec{x}-\vec{u}\}$ are dependent.

$$c_1(\vec{u}-\vec{v}) + c_2(\vec{v}-\vec{w}) + c_3(\vec{w}-\vec{x}) + c_4(\vec{x}-\vec{u}) = \vec{0}$$

is satisfied if $c_1 = 1, c_2 = 1, c_3 = 1, c_4 = 1$

$$\Rightarrow \vec{u} - \vec{v} = -1(\vec{v}-\vec{w}) - 1(\vec{w}-\vec{x}) - 1(\vec{x}-\vec{u}).$$

Example:

Are the polynomials $\{x^2+x, x^2-1, x+1\}$ linearly independent in $P_2(\mathbb{R})$.

Suppose $c_1(x^2+x)+c_2(x^2-1)+c_3(x+1)=0$

$$\Rightarrow (c_1+c_2)x^2+(c_1+c_3)x+(-c_2+c_3)=0$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right] -R1 \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right] -R3 \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore, there are an infinite number of nonzero solutions

$\Rightarrow$ linearly dependent.

Definition: A set of vectors $\vec{v}_1, \dots, \vec{v}_n \in V$ is a basis for V if

1. $\text{span}\{\vec{v}_1, \dots, \vec{v}_n\} = V$
2. $\vec{v}_1, \dots, \vec{v}_n$ are linearly independent

The dimension of V is $\dim(V)=n$.

Example:

Find a basis for

$$V = \text{span}\left(\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & -3 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -2 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ -1 & -2 \end{bmatrix} \right)$$

Consider

$$c_1 \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 & -3 \\ 1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 0 \\ -2 & 2 \end{bmatrix} + c_4 \begin{bmatrix} 0 & 2 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} c_1+2c_2+c_3 &= 0 \\ -c_1-3c_2+2c_4 &= 0 \\ c_2-2c_3-c_4 &= 0 \\ 2c_3-2c_4 &= 0 \end{aligned}$$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 0 \\ -1 & -3 & 0 & 2 & 0 \\ 0 & 1 & -2 & -1 & 0 \\ 0 & 0 & 2 & -2 & 0 \end{array}\right] \begin{array}{l} \\ +R1 \\ \\ \\ \end{array} \Rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 1 & 0 & 0 \\ 0 & -1 & 1 & 2 & 0 \\ 0 & 1 & -2 & -1 & 0 \\ 0 & 0 & 2 & -2 & 0 \end{array}\right] \updownarrow \Rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -2 & -1 & 0 \\ 0 & -1 & 1 & 2 & 0 \\ 0 & 0 & 2 & -2 & 0 \end{array}\right] \begin{array}{l} \\ \\ +R2 \\ \\ \end{array}$$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -2 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & -2 & 0 \end{array}\right] \begin{array}{l} \\ \\ \\ +2R3 \end{array} \Rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -2 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore, a basis for V is

$$V = \text{span}\left\{ \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & -3 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -2 & 2 \end{bmatrix} \right\}$$