Lecture #6: Linear Transformations

Definition - If $T: V \to W$ is a function from a vector space V into a vector space W, then T is called a linear transformation from V to W if for all $\vec{u}, \vec{v} \in V$ and $c \in F$

$$T[\vec{u} + c\vec{v}] = T[u] + cT[\vec{v}].$$

Example:

$T: P_2(\mathbb{R}) \to P_1(\mathbb{R})$ defined by

$$T[c_0 + c_1 x + c_2 x^2] = c_1 + 2c_2 x$$

is a linear transformation.

proof:

Let $\vec{u}, \vec{v} \in P_2(\mathbb{R})$ and $\lambda \in F$. Then there exists $c_0, c_1, c_2, d_0, d_1, d_2 \in \mathbb{R}$ such that

$$\vec{u} = c_0 + c_1 x + c_2 x^2, \quad \vec{v} = d_0 + d_1 x + d_2 x^2$$

$$\begin{aligned}
\Rightarrow T[\vec{u} + \lambda\vec{v}] &= T[c_0 + c_1 x + c_2 x^2 + \lambda(d_0 + d_1 x + d_2 x^2)] \\
&= T[c_0 + \lambda d_0 + (c_1 + \lambda d_1)x + (c_2 + \lambda d_2)x^2] \\
&= c_1 + \lambda d_1 + 2(c_2 + \lambda d_2)x \\
&= c_1 + 2c_2 x + \lambda(d_1 + 2d_2 x) \\
&= T[\vec{u}] + \lambda T[\vec{v}].
\end{aligned}$$

Therefore, T is a linear transformation.

Coordinates

If $\beta = \{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis for a vector space V and $\vec{U} \in \text{span}\{\vec{v}_1, \ldots, \vec{v}_n\}$ then there exists $c_1, \ldots, c_n \in F$ such that

$$\vec{U} = c_1\vec{v}_1 + \ldots + c_n\vec{v}_n$$

Define a linear transformation $[\vec{U}]_\beta : V \to F^n$ by

$$[\vec{U}]_\beta = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

The values $c_1, \ldots, c_n$ are called the coordinates of $\vec{U}$ with respect to the β-basis.

Previous Example:

$$T[c_0 + c_1 x + c_2 x^2] = c_1 + 2c_2 x, \text{ i.e. } T[\vec{U}] = \vec{V}.$$

A basis for $P_2(\mathbb{R})$ and $P_1(\mathbb{R})$ are given by

$$A = \{1, x, x^2\}, \quad \beta = \{1, x\}$$

Therefore,

$$[\vec{U}]_A = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} \quad [\vec{V}]_\beta = \begin{bmatrix} c_1 \\ 2c_2 \end{bmatrix}$$

In matrix notation

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}}_{[T(\beta, A)]} \underbrace{\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}}_{[\vec{U}]_A} = \underbrace{\begin{bmatrix} c_1 \\ 2c_2 \end{bmatrix}}_{[\vec{V}]_\beta}$$

A matches

Observation:

Let $T : V \to W$ be a linear transformation and $A = \{\vec{v}_1, \ldots, \vec{v}_n\}$ a basis for V. Therefore, if

$$\vec{v} = c_1\vec{v}_1 + \ldots + c_n\vec{v}_n$$

then

$$T[\vec{v}] = T[c_1\vec{v}_1 + \ldots + c_n\vec{v}_n] = c_1 T[\vec{v}_1] + \ldots + c_n T[\vec{v}_n].$$

To determine a linear transformation, only need to know $T[\vec{v}_1], \ldots, T[v_n]$!

$$\Rightarrow \operatorname{im}(T) = \{\vec{w} \in W : \text{there exists } \vec{v} \text{ such that } T[\vec{v}] = \vec{w}\}$$
$$= \operatorname{span}\{T[\vec{v}_1], \ldots, T[\vec{v}_n]\}.$$

Example:

The set $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, where

$$\vec{v}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}, \vec{v}_2 = \begin{bmatrix}1\\1\\0\end{bmatrix}, \vec{v}_3 = \begin{bmatrix}1\\0\\0\end{bmatrix}.$$

is a basis for $\mathbb{R}^3$. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by

$$T[\vec{v}_1] = \begin{bmatrix}1\\0\end{bmatrix}, T[\vec{v}_2] = \begin{bmatrix}2\\-1\end{bmatrix}, T[\vec{v}_3] = \begin{bmatrix}4\\5\end{bmatrix}$$

Find a formula for $T\begin{bmatrix}x\\y\\z\end{bmatrix}$ and compute:

$$T\begin{bmatrix}2\\-3\\5\end{bmatrix}.$$

We first express $\begin{bmatrix}x\\y\\z\end{bmatrix}$ as a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \begin{bmatrix}x\\y\\z\end{bmatrix}$$

$$\begin{aligned} c_1 + c_2 + c_3 &= x \\ c_1 + c_2 &= y \\ c_1 &= z \end{aligned} \quad \Rightarrow c_1 = z, \ c_2 = y - z, \ c_3 = x - y.$$

Therefore,

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = z\begin{bmatrix}1\\1\\1\end{bmatrix} + (y-z)\begin{bmatrix}1\\1\\0\end{bmatrix} + (x-y)\begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$\Rightarrow T\begin{bmatrix}x\\y\\z\end{bmatrix} = z\begin{bmatrix}1\\0\end{bmatrix} + (y-z)\begin{bmatrix}2\\-1\end{bmatrix} + (x-y)\begin{bmatrix}4\\5\end{bmatrix}$$

$$\Rightarrow T\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}4x-2y-z\\3x-4y+z\end{bmatrix} \Rightarrow T\begin{bmatrix}2\\-3\\5\end{bmatrix} = \begin{bmatrix}9\\23\end{bmatrix}.$$

Matrix Multiplication Revisited

The standard basis in $\mathbb{F}^n$ is

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, \vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

Suppose $\vec{v} \in \mathbb{F}^n$, $\vec{w} \in \mathbb{F}^n$, $A \in M_{m \times n}(\mathbb{F})$ and

$$\vec{w} = A\vec{v}$$

$$\Rightarrow \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

$$\Rightarrow w_i = \sum_{j=1}^{n} a_{ij} \underbrace{v_j}_{\text{match}}$$ (Very useful relationship for proofs)

Therefore, what is $A\vec{e}_1, A\vec{e}_2, \dots, A\vec{e}_n$?

$$\underline{A\vec{e}_1}, \dots, \underline{A\vec{e}_n}$$

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} \qquad \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$$\text{Column } \#1 \qquad \text{Column } \#n$$

$$\boxed{\text{In the matrix representation, the columns are the images of the basis vectors}}$$

Example:

If $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4x-2y-z \\ 3x-4y+z \end{bmatrix}$. Consequently,

$$T\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}, T\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \end{bmatrix}, T\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\Rightarrow T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 4 & -2 & -1 \\ 3 & -4 & 1 \end{bmatrix}}_{\text{Matrix Represention}} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Example:

Let $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be defined by

$$T[c_0 + c_1 x + c_2 x^2] = c_0 + c_1(3x-5) + c_2(3x-5)^2$$

The standard basis for P_2 is $\{1, x, x^2\} = \beta$.

$$\Rightarrow [c_0 + c_1 x + c_2 x^2]_\beta = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}$$

$$\Rightarrow [1]_\beta = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ [x]_\beta = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ [x^2]_\beta = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Therefore,

$$\begin{aligned}
[T[c_0 + c_1 x + c_2 x^2]]_\beta &= c_0[T[1]]_\beta + c_1[T[x]]_\beta + c_2[T[x^2]]_\beta \\
&= c_0[1]_\beta + c_1[3x-5]_\beta + c_2[(3x-5)^2]_\beta \\
&= c_0\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_1\begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 25 \\ -30 \\ 9 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -5 & 25 \\ 0 & 3 & -30 \\ 0 & 0 & 9 \end{bmatrix}\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} \\
&= \left[[T[1]]_\beta \,\vdots\, [T[x]]_\beta \,\vdots\, [T[x^2]]_\beta \right]\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} \\
&= \underbrace{[T(\underset{\text{input basis}}{\beta}, \underset{\text{output basis}}{\beta})]}_{\text{matrix representation of linear transformation}} \cdot \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}
\end{aligned}$$