Lecture #7: Coordinates and change of basis

Example:

Let $S_1 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$, $\beta_1 = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}$,

$S_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$, $\beta_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$,

$T: \mathbb{Z}_3^3 \to \mathbb{Z}_3^2$ be defined by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} y+z \\ 2x+y \end{bmatrix}$$

1. Let $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. What is $[\vec{v}]_{\beta_1}$?

$\Rightarrow \vec{v} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

coordinates

$\Rightarrow c_1 = 1,\ c_2 = 0$

$\Rightarrow [\vec{v}]_{\beta_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

2. Let $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. What is $[\vec{v}]_{\beta_1}$?

$v = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$$\to \left[\begin{array}{cc|c} 2 & 2 & 1 \\ 1 & 2 & 1 \end{array}\right] \begin{array}{l} \times 2 \\ \end{array}, \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & 2 & 1 \end{array}\right] \begin{array}{l} \\ +2R1 \end{array} \to \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 2 \end{array}\right] \begin{array}{l} +2R2 \\ \end{array} \to \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 2 \end{array}\right]$$

$\Rightarrow c_1 = 0,\ c_2 = 2$

$\Rightarrow [\vec{v}]_{\beta_1} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$.

2. What is $[T(S_2, S_1)]$?

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}0\\2\end{bmatrix},\ T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}1\\1\end{bmatrix},\ T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}1\\0\end{bmatrix}$$

$$\Rightarrow [T(S_2, S_1)] = \begin{bmatrix}0 & 1 & 1\\ 2 & 1 & 0\end{bmatrix}.$$

3. What is $[T(\beta_2, S_1)]$?

$$T\left(\begin{bmatrix}1\\0\\2\end{bmatrix}\right)=\begin{bmatrix}2\\2\end{bmatrix},\ T\left(\begin{bmatrix}2\\2\\2\end{bmatrix}\right)=\begin{bmatrix}1\\0\end{bmatrix},\ T\left(\begin{bmatrix}0\\1\\2\end{bmatrix}\right)=\begin{bmatrix}0\\1\end{bmatrix}$$

$$\Rightarrow [T(\beta_2, S_1)] = \begin{bmatrix}2 & 1 & 0\\ 2 & 0 & 1\end{bmatrix}$$

4. What is $[T(\beta_2, \beta_1)]$?

$$[T(\beta_2, \beta_1)] = \left[\begin{bmatrix}2\\2\end{bmatrix}_{\beta_1} \vdots \begin{bmatrix}1\\0\end{bmatrix}_{\beta_1} \vdots \begin{bmatrix}0\\1\end{bmatrix}_{\beta_1}\right]$$

$$\begin{bmatrix}2\\2\end{bmatrix} = c_1\begin{bmatrix}2\\1\end{bmatrix} + c_2\begin{bmatrix}2\\2\end{bmatrix} = \begin{bmatrix}2 & 2\\ 1 & 2\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}$$

Coordinates

We can solve for all coordinates all at once:

$$\left[\begin{array}{cc:ccc}2 & 2 & 2 & 1 & 0\\ 1 & 2 & 2 & 0 & 1\end{array}\right]\times 2 \to \left[\begin{array}{cc:ccc}1 & 1 & 1 & 2 & 0\\ 1 & 2 & 2 & 0 & 1\end{array}\right]{+2R1} \to \left[\begin{array}{cc:ccc}1 & 1 & 1 & 2 & 0\\ 0 & 1 & 1 & 1 & 1\end{array}\right]{+2R2}$$

$$\to \left[\begin{array}{cc:ccc}1 & 0 & 0 & 1 & 2\\ 0 & 1 & 1 & 1 & 1\end{array}\right]$$

Therefore,

$$\begin{bmatrix}2\\2\end{bmatrix}_{\beta_1} = \begin{bmatrix}0\\1\end{bmatrix},\ \begin{bmatrix}1\\0\end{bmatrix}_{\beta_1} = \begin{bmatrix}1\\1\end{bmatrix},\ \begin{bmatrix}0\\1\end{bmatrix}_{\beta_1} = \begin{bmatrix}2\\1\end{bmatrix}$$

$$\Rightarrow [T(\beta_2, \beta_1)] = \begin{bmatrix}0 & 1 & 2\\ 1 & 1 & 1\end{bmatrix}$$

Change of Basis Example:

Let $V = \mathbb{Z}_3^2$ and

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \beta = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

1. What is $P = [I(\beta, S)]$?

$$I\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad I\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\Rightarrow P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \leftarrow$$ Changes vectors from β basis representation to S basis representation.

2. What is $Q = [I(S, \beta)]$?

$$I\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}_\beta = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$I\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}_\beta = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\Rightarrow Q = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.$$

3. If $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x \end{bmatrix}$. What is $[T(S_1, S_2)]$?

$$\Rightarrow T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow [T(S, S)] = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

4. What is $[T(\beta,\beta)]$?

Let $D=[T(\beta,\beta)]$, $\tilde{T}=[T(S,S)]$

Therefore, for all $\vec{v}\in\mathbb{Z}_3^2$

$$D\cdot[\vec{v}]_\beta = I[(S,\beta)]\,\tilde{T}\,I[(\beta,S)]\,[\vec{v}]_\beta$$

Change from S to β

Standard transformation.

Change from β to S

$$\Rightarrow \boxed{D=Q\tilde{T}P=P^{-1}\tilde{T}P,}$$

Since $I[(S,\beta)]=I[(\beta,S)]^{-1}$

Two matrices A,B are similar if there exists a matrix P such that

$$\boxed{A=P^{-1}BP}$$

Similar matrices are essentially the same matrix, just in different coordinate systems.