Lecture #8: Nullity, Rank, and Isomorphisms

Definition - Let $T: V \to W$ be a linear transformation
The nullspace or kernel of T is defined by
$$\ker(T) = \{\vec{v} \in V : T(\vec{v}) = \vec{0}\}.$$
The image of T is defined by
$$\operatorname{im}(T) = \{\vec{w} \in W : T(\vec{v}) = \vec{w} \text{ for some } \vec{w}\}.$$

Example:
$T: M_{2\times 2}(\mathbb{R}) \to P_2(\mathbb{R})$ defined by
$$T\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = (a_{11} + a_{12} + a_{21} + a_{22}) + (a_{11} + a_{12} + a_{21} + a_{22})t$$

- $T(\vec{v}) = \vec{0}$

$$\Rightarrow a_{11} + a_{12} + a_{21} + a_{22} = 0$$
$$\Rightarrow a_{11} = -a_{12} - a_{21} - a_{22}$$
$$\Rightarrow \vec{v} = \begin{bmatrix} -a_{12} - a_{21} - a_{22} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{12}\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} + a_{21}\begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} + a_{22}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\Rightarrow \ker(T) = \operatorname{span}\left(\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

- The nullity of T, denoted $\operatorname{nul}(T)$, is given by
$$\operatorname{nullity}(T) = \dim(\ker(T))$$
In this problem
$$\operatorname{nullity}(T) = 3.$$

- Suppose $\vec{w} \in \operatorname{im}(T)$. Therefore, there exists $a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{R}$ such that
$$\vec{w} = (a_{11} + a_{12} + a_{21} + a_{22}) + (a_{11} + a_{12} + a_{21} + a_{22})t$$
$$= a_{11}(1+t) + a_{21}(1+t) + a_{12}(1+t) + a_{22}(1+t)$$
$$\Rightarrow \operatorname{im}(T) = \operatorname{span}\{1+t, 1+t, 1+t, 1+t\}$$
$$= \operatorname{span}\{1+t\}.$$

- The rank of T is the dimension of T. In this case
$$\text{rank}(T)=1.$$

Theorem - Let $T: V \to W$ be a linear transformation. Then,
$$\dim(V)=\text{nullity}(T)+\text{rank}(T)$$

~~proof~~:

Let $\beta=\{\vec{v}_1,\dots,\vec{v}_n\}$ be a basis for $\ker(T)$. Expand β to a basis $\mathcal{A}=\{\vec{v}_1,\dots,\vec{v}_n,\vec{w}_1,\dots,\vec{w}_m\}$ for V. Therefore, for all $\vec{v}\in V$, there exists $a_1,\dots,a_n\in F$, $b_1,\dots,b_m\in F$ such that
$$\vec{v}=a_1\vec{v}_1+\dots+a_n\vec{v}_n+b_1\vec{w}_1+\dots+b_m\vec{w}_m$$
$$\Rightarrow T[\vec{v}]=b_1T[\vec{w}_1]+\dots+b_mT[\vec{w}_m].$$
$$\Rightarrow \text{im}(T)=\text{span}\{T[w_1],\dots,T[\vec{w}_m]\}.$$
Now, suppose there exists $c_1,\dots,c_m\in F$ such that
$$c_1T[\vec{w}_1]+\dots+c_mT[w_m]=0$$
$$\Rightarrow T[c_1\vec{w}_1+\dots+c_m\vec{w}_m]=0$$
$$\Rightarrow c_1\vec{w}_1+\dots+c_m\vec{w}_m\in\ker(T)$$
But we assumed $\text{span}\{\vec{w}_1,\dots,\vec{w}_m\}\cap\ker(T)=0$
$$\Rightarrow c_1\vec{w}_1+\dots+c_m\vec{w}_m=0$$
$$\Rightarrow c_1=\dots=c_m=0,$$
since $\vec{w}_1,\dots,\vec{w}_m$ are linearly independent.

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Theorem – A linear transformation is one-to-one if and only if $\ker(T) = 0$.

~~proof~~:

(i) Suppose T is one-to-one and $\vec{u} \in \ker(T)$. Therefore, $T[\vec{u}] = \vec{0}$. However, since $T[\vec{0}] = \vec{0}$ it follows that $\vec{u} = \vec{0}$.

(ii) Suppose $\ker(T) = \vec{0}$ and there exists $\vec{u}_1, \vec{u}_2 \in V$ such that $T[\vec{u}_1] = T[\vec{u}_2]$. Therefore, $T[\vec{u}_1 - \vec{u}_2] = 0 \Rightarrow \vec{u}_1 - \vec{u}_2 = \vec{0} \Rightarrow \vec{u}_1 = \vec{u}_2$.

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Corollary – Let $T: V \to W$ be a linear transformation with $\dim(V) = \dim(W)$. Then, T is one-to-one if and only if T maps onto W.

~~proof~~

T is one-to-one if and only if $\ker(T) = 0$. Therefore,

$$\text{nullity} = 0 \Leftrightarrow \text{rank}(T) = \dim(V) = \dim(W)$$

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Theorem – $V \cong W$ if and only if $\dim(V) = \dim(W) < \infty$.

~~proof~~:

Let $A = \{\vec{v}_1, \ldots, \vec{v}_n\}$, $B = \{\vec{w}_1, \ldots, \vec{w}_n\}$ be bases for V and W. Define $T: A \to B$ by $T[v_i] = \vec{w}_i$. By linear independence $\ker(T) = 0$ and thus T is one-to-one and onto.

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