Lecture #9: Diagonalization and Eigenvalues

Example:

Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$T(\vec{v}) = A\vec{v} = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}\vec{v}.$$

Consider the basis

$$\beta = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\} = \{\vec{u}_1, \vec{u}_2\}, \quad S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} = \{\vec{e}_1, \vec{e}_2\}.$$

$$\Rightarrow T(\vec{u}_1) = A\vec{u}_1 = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} = -2\begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

$$T(\vec{u}_2) = A\vec{u}_2 = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 15 \\ 20 \end{bmatrix} = 5\begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

There

$$[T(\beta, \beta)] = \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix} = \Lambda$$

$$\Rightarrow \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}\vec{v} = \underset{A}{[T(S,S)]}\vec{v} = \underset{P^{-1}}{[I(\beta,S)]}\,\underset{\Lambda}{[T(\beta,\beta)]}\,\underset{P}{[I(S,\beta)]}\vec{v}$$

$$\Rightarrow \boxed{A = P^{-1}\Lambda P}$$

$$P = \begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix}, \quad P^{-1} = \frac{1}{7}\begin{bmatrix} 4 & 3 \\ -1 & 1 \end{bmatrix}$$

$$\Rightarrow \boxed{\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 4 & -3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix}} \longrightarrow \text{Diagonalization}$$

Example:

If $A=\begin{bmatrix}1 & 3\\ 4 & 2\end{bmatrix}$, what is A^3?

$$A=P^{-1}\Delta P$$

$$\Rightarrow A^3=P^{-1}\Delta PP^{-1}\Delta P\cdot P^{-1}\Delta P=P^{-1}\Delta^3 P$$

Therefore,

$$A^3=\frac{1}{7}\begin{bmatrix}4 & -3\\ 1 & 1\end{bmatrix}\begin{bmatrix}-8 & 0\\ 0 & 125\end{bmatrix}\begin{bmatrix}1 & 3\\ -1 & 4\end{bmatrix}.$$

Remark:

The key to diagonalization was the existence of a vector (eigenvector) and scalar $\lambda\in F$ (eigenvalue) such that

$$T(\vec{v})=\lambda\vec{v},\ [T(S,S)]\vec{v}=\lambda\vec{v}\ \text{(If a matrix exists).}$$

If $\beta=\{\vec{v}_1,\dots,\vec{v}_n\}$ is a basis of eigenvectors then

$$[T(\beta,\beta)]=\begin{bmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{bmatrix}$$

Example:

If $T:\mathcal{C}^\infty(\mathbb{R})\to\mathcal{C}^\infty(\mathbb{R})$ is defined by $T(f)=\frac{df}{dx}$, what are the eigenvectors of T?

$$T(f)=\lambda f$$

$$\Rightarrow \frac{df}{dx}=\lambda f$$

For every real number $\lambda\in\mathbb{R}$ there is a coorresponding eigenfunction

$$f_\lambda(x)=e^{\lambda x}.$$