MTH 352/652
Spring 2025
Exam 2
02/19/25

Name (Print): Key

This exam contains 9 pages (including this cover page) and 9 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Short answer questions:** Questions labeled as **"Short Answer"** can be answered by simply writing an equation or a sentence or appropriately drawing a figure. No calculations are necessary or expected for these problems.
- **Unless the question is specified as short answer, mysterious or unsupported answers might not receive full credit.** An incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

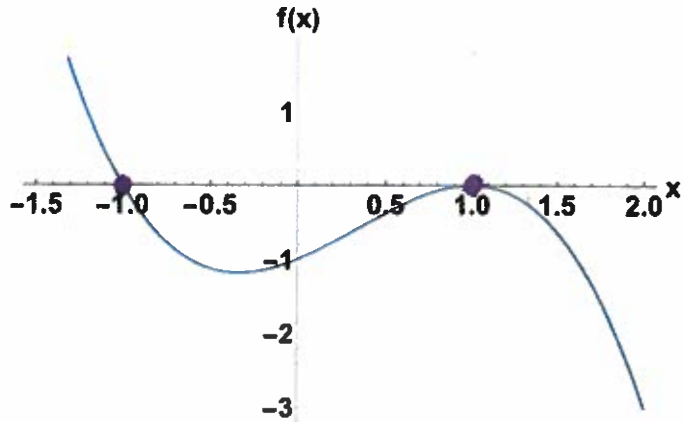
Do not write in the table to the right.

Problem	Points	Score
1	15	
2	10	
3	5	
4	15	
5	10	
6	10	
7	10	
8	15	
9	10	
Total:	100	

1. (15 points) Consider the differential equation

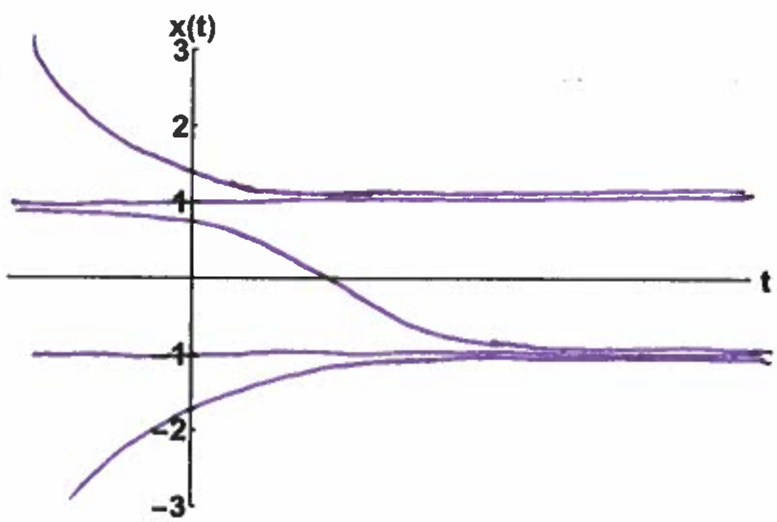
$$\frac{dx}{dt} = f(x),$$

where $f(x)$ is plotted below.



(a) (5 points) On the graph, circle the x-coordinate of any fixed points.

(b) (10 points) Sketch the solution curves to this differential equation on the set of axes provided below. Be sure to plot enough curves to illustrate the full behavior of the system. **Hint:** At a minimum, this is five curves.



2. (10 points) Find the solution to the following partial differential equation on the domain $x \in \mathbb{R}$ and $t \geq 0$:

$$\frac{\partial u}{\partial t} = u\cos(t),$$
$$u(x,0) = f(x),$$

where $f(x)$ is an arbitrary smooth function.

$$\frac{\partial u}{\partial t} = u\cos(t)$$

$$\Rightarrow \int_{f(x)}^{u} \frac{1}{v}\,dv = \int_0^t \cos(s)\,ds$$

$$\Rightarrow \ln(u) - \ln(f(x)) = \sin(t)$$

$$\Rightarrow u(x,t) = f(x)e^{\sin(t)}$$

3. (5 points) **Short Answer:** Classify the following differential equations as either homogenous linear, inhomogeneous linear, or nonlinear. Just check the appropriate box, no work needs to be shown.

(a) $u_t + uu_x = 0$

- ☐ Homogenous Linear
- ☐ Inhomogenous Linear
- ☑ Nonlinear

(b) $\cos(t)u_t + \sin(x)u_x = 0$

- ☑ Homogenous Linear
- ☐ Inhomogenous Linear
- ☐ Nonlinear

(c) $u_{tt} + u_{xx} = \sin(u)$

- ☐ Homogenous Linear
- ☐ Inhomogenous Linear
- ☑ Nonlinear

(d) $u_t = u_{xx} + \sin(x)$

- ☐ Homogenous Linear
- ☑ Inhomogenous Linear
- ☐ Nonlinear

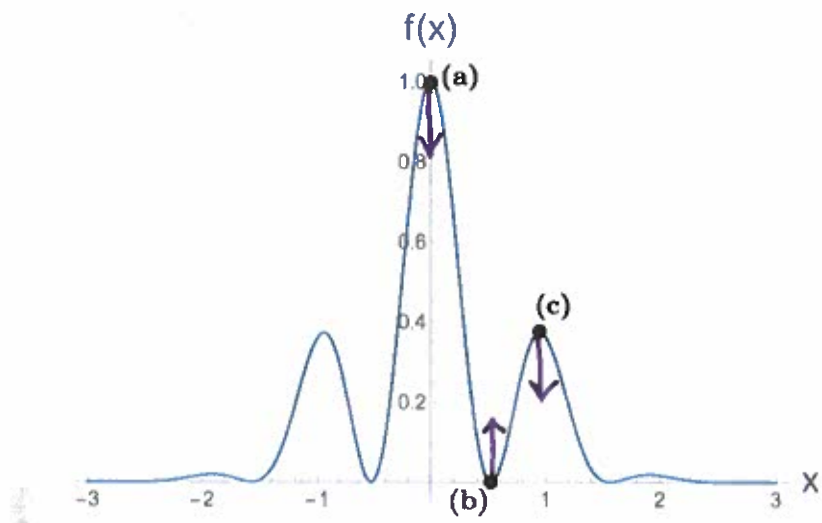
(e) $u_t + uu_x = 0$

- ☐ Homogenous Linear
- ☐ Inhomogenous Linear
- ☑ Nonlinear

4. (15 points) Suppose $u(x,t)$ solves the following initial value problem:

$$u_t = u_{xx},$$
$$u(x,0) = f(x),$$

where $f(x)$ is plotted below.



(a) (5 points) **Short Answer:** At points **(a)**, **(b)**, and **(c)** indicate on the figure whether $u(x,t)$ is increasing or decreasing in time at $t = 0$.

(a) Decreasing
(b) Increasing
(c) Decreasing

(b) (5 points) **Short Answer:** Given that $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)\,dx = \frac{(1+e^4)\sqrt{\pi}}{2e^4}$, what is $\lim_{t\to\infty} u(x,t)$ for all $x \in \mathbb{R}$.

$$\lim_{t\to\infty} u(x,t) = 0$$

(c) (5 points) **Short Answer:** Given that $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)\,dx = \frac{(1+e^4)\sqrt{\pi}}{2e^4}$, what is $\lim_{t\to\infty} \int_{-\infty}^{\infty} u(x,t)\,dx$.

$$\lim_{t\to\infty} \int_{-\infty}^{\infty} u(x,t)\,dx = \frac{(1+e^4)\sqrt{\pi}}{2e^4}$$

5. (10 points) Consider the following partial differential equation on $\mathbb{R}$:

$$au_x + bu_y = 0,$$

where $a, b \in \mathbb{R}$. By changing variables to $X = ax + by$ and $Y = bx - ay$, show that if $a, b \neq 0$ then this equation can be expressed in the form:

$$u_X = 0.$$

Let $X = ax+by$, $Y = bx-ay$. Therefore,

$$\frac{\partial}{\partial x} = \frac{\partial X}{\partial x}\frac{\partial}{\partial X} + \frac{\partial Y}{\partial x}\frac{\partial}{\partial Y} = a\frac{\partial}{\partial X} + b\frac{\partial}{\partial Y}$$

$$\frac{\partial}{\partial y} = \frac{\partial X}{\partial y}\frac{\partial}{\partial X} + \frac{\partial Y}{\partial y}\frac{\partial}{\partial Y} = b\frac{\partial}{\partial X} - a\frac{\partial}{\partial Y}$$

Therefore,

$$0 = au_x + bu_y = a^2 u_X + ab\,u_Y + b^2 u_X - ab\,u_Y$$

$$\Rightarrow (a^2+b^2)u_X = 0$$

$$\Rightarrow u_X = 0.$$

6. (10 points) **Short Answer:** Suppose $u(x,t)$ solves the following initial value problem:

$$u_t = u + \delta u_{xx} - u_{xxxx},\ x \in \mathbb{R},\ t > 0$$
$$u(x,0) = f(x),$$

where $\delta > 0$ is a constant. Using Fourier transforms, write down an initial value problem satisfied by $\hat{u}(k,t)$. Be sure to include the initial conditions satisfied at $\hat{u}(k,0)$. **Do not solve this equation!**

$$\hat{U}_t = \hat{U} - \delta K^2 \hat{U} + K^4 \hat{U}$$
$$\hat{U}(K,0) = \hat{f}(K)$$

7. (10 points) Given that

$$\mathcal{F}\left[x^2 e^{-|x|}\right](k) = \frac{4(1-3k^2)}{(1+k^2)^3},$$

calculate $\mathcal{F}\left[\frac{1-3x^2}{(1+x^2)^3}\right](k)$.

$$\mathcal{F}\left[\frac{1-3x^2}{(1+x^2)^3}\right] = \int_{-\infty}^{\infty} \frac{1-3x^2}{(1+x^2)^3} e^{ikx} dx$$
$$= \frac{2\pi}{4} \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{4(1-3x^2)}{(1+x^2)^3} e^{ikx} dx$$
$$= \frac{\pi}{2} \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{4(1-3u^2)}{(1+u^2)^3} e^{-iku} du$$
$$= \frac{\pi}{2} K^2 e^{-|K|}.$$

8. (15 points) Suppose that $u(x,t)$ is Schwartz class function that satisfies the following partial differential equation on $\mathbb{R}$:

$$u_t = -u_{xxxx}.$$

(a) (5 points) Show that the total mass $M(t) = \int_{-\infty}^{\infty} u(x,t)dx$ is constant in time.

$$\frac{d}{dt}\int_{-\infty}^{\infty} u\,dx = \int_{-\infty}^{\infty} u_t\,dx$$

$$= -\int_{-\infty}^{\infty} u_{xxxx}\,dx$$

$$= -u_{xxx}\Big|_{-\infty}^{\infty}$$

$$= 0$$

(b) (10 points) Show that the total energy $E(t) = \int_{-\infty}^{\infty} u(x,t)^2 dx$ is decreasing in time.

$$\frac{d}{dt}\int_{-\infty}^{\infty} u^2\,dx = \int_{-\infty}^{\infty} 2uu_t\,dx$$

$$= -2\int_{-\infty}^{\infty} uu_{xxxx}\,dx$$

$$= -2\left(\cancelto{0}{uu_{xxx}}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u_x u_{xxx}\,dx\right)$$

$$= 2\int_{-\infty}^{\infty} u_x u_{xxx}\,dx$$

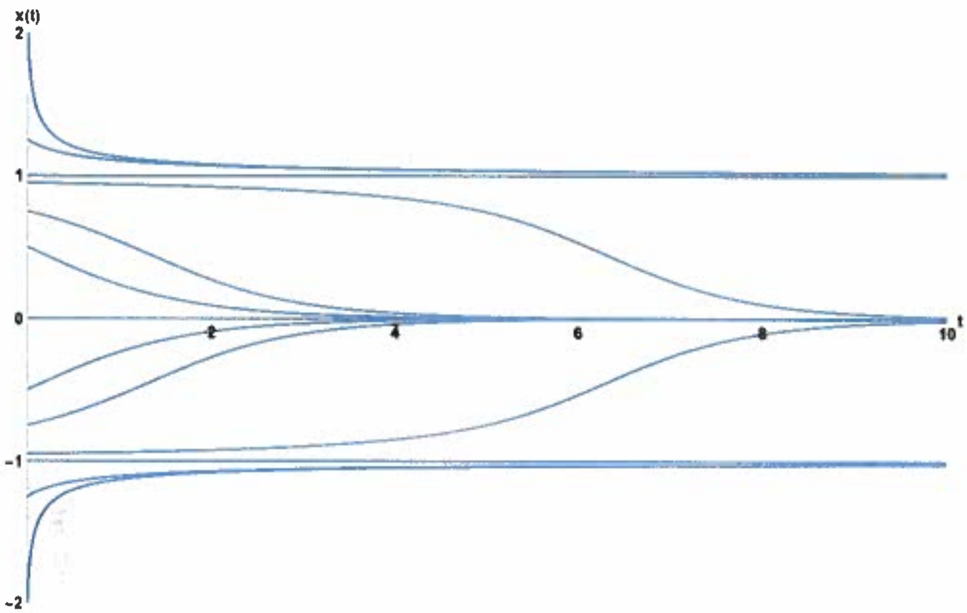
$$= 2\left(\cancelto{0}{u_x u_{xx}}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u_{xx}^2\,dx\right)$$

$$= -2\int_{-\infty}^{\infty} u_{xx}^2\,dx$$

$$\leq 0.$$

9. (10 points) The figure below is a plot of the characteristic curves for the following initial value problem

$$u_t + c(x)u_x = 0$$
$$u(x,0) = \frac{x}{1+x^2}.$$



If $u(x,t)$ is a solution to this PDE, compute

$$u^*(x) = \lim_{t\to\infty} u(x,t).$$

$$U^*(x) = \begin{cases} 0, & |x|>1 \\ -\frac{1}{2}, & -1 \le x < 0 \\ 0, & x = 0 \\ \frac{1}{2}, & 0 < x \le 1 \end{cases}$$