MTH 352/652
Homework #1

Due Date: January 24, 2025

3/2 1. Solve the following separable ordinary differential equations assuming the initial condition $x(0) = x_0$:

(a) $\dfrac{dx}{dt} = e^{x+t}$

(b) $\dfrac{dx}{dt} = tx + x + t + 1$

(c) $\dfrac{dx}{dt} = \dfrac{t^2+2}{x}$

2 2. Consider the following initial value problem

$$\begin{cases} \dfrac{dy}{dt} - a(t)y = 0 \\ y(0) = y_0 \end{cases},$$

where $a(t)$ is a smooth real valued function and $y_0 \in \mathbb{R}$.

(a) Show that the solution $y(t)$ is positive for all t, negative for all t, or zero for all t.

(b) If $y_0 = 1$, find a function $a(t)$ such that the solution $y(t)$ has $y(1) = 2$.

1/2 3. Assuming $y(0) = y_0$, find the solutions to the following ordinary differential equations by making the indicated change of variables and solving the resulting simpler differential equation:

(a) $\dfrac{dy}{dt} = e^{t-y} - e^t$, (let $u = e^y$),

(b) $t\dfrac{dy}{dt} = y(\ln(ty) - 1)$, (let $u = ty$),

(c) $2ty\dfrac{dy}{dt} = y^2 - t$, (let $u = y^2$).

4. A human body after death cools at a rate that is nearly proportional to the difference between the body and the surroundings. Assume that this cooling rate is $-\beta(T(t) - T_0)$ where $T(t)$ is the temperature of the body as a function of time, T_0 is the temperature of the surroundings, and β is an empirical constant. Suppose that a body discovered at 11:00 AM has a temperature reading of $97.8^\circ F$ and at 1:00 PM has a temperature reading of $96.2^\circ F$. The room temperature was constant at 86.4°.

   (a) Write down an ordinary differential equation for the temperature of the body. Be sure to include your initial conditions.

   (b) By solving your model, determine the value of β.

   (c) Assuming that at the time of death the body temperature was $98.6^\circ F$, determine an approximate time of death.

2 5. Consider the following initial value problem:

$$\begin{cases} \dfrac{dx}{dt} = -x^{1/3} \\ x(0) = 0 \end{cases} . \tag{1}$$

   (a) For $t_0 < 0$, show that the function

$$x(t) = \begin{cases} \left(\frac{2}{3}\right)^{3/2} (t_0 - t)^{3/2} & t < t_0 \\ 0 & t \geq t_0 \end{cases},$$

   is a differentiable function with a continuous derivative.

   (b) Show that for $t_0 < 0$ the function defined in part (a) is a solution to the initial value problem.

   (c) Sketch $x(t)$ for $t_0 = -1, -2, -3$.

   (d) What does the existence of this family of solutions tell you about the uniqueness of solution to Equation (1)? Why does this not contradict the existence and uniqueness theorem of ordinary differential equations.

2 6. For the following differential equations, sketch the solution curves $x(t)$ for different initial conditions. You must plot enough curves to illustrate the behavior of the solutions.

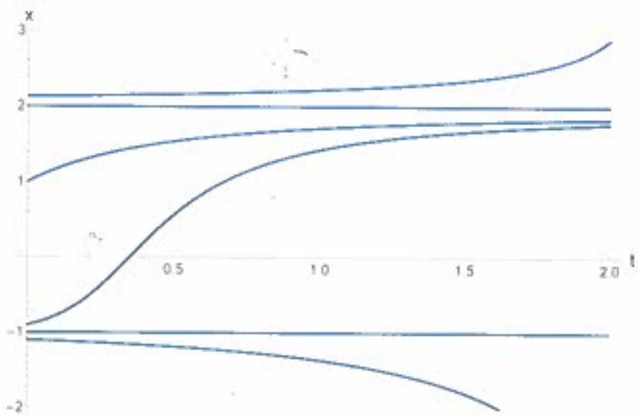
   (a) $\dfrac{dx}{dt} = 4x^2 - 16$

   (b) $\dfrac{dx}{dt} = 1 - x^{14}$

   (c) $\dfrac{dx}{dt} = x - x^3$

   (d) $\dfrac{dx}{dt} = e^{-x}\sin(x)$.

7. The curves $x(t)$ illustrated below correspond to solution curves for the differential equation $\frac{dx}{dt} = f(x)$.



(a) Sketch a graph of $f(x)$ that is consistent with this figure.

(b) Give a formula for $f(x)$ that is consistent with this figure.

8. For each of parts (a)-(e), find an equation $\frac{dx}{dt} = f(x)$ with the stated properties, or if there are no examples, explain why not.

(a) Every real number is a fixed point.

(b) Every integer is a fixed point, and there are no others.

(c) There are precisely three fixed points, and there are no others.

(d) There are no fixed points.

(e) There are precisely 100 fixed points.

Homework #1

#1

Solve the following seperable ODEs assuming the initial condition $x(0)=x_0$

(a) $\frac{dx}{dt} = e^{x+t}$

(b) $\frac{dx}{dt} = tx + x + t + 1$

(c) $\frac{dx}{dt} = \frac{t^2+2}{x}$

Solution:

(a) $\frac{dx}{dt} = e^{x+t} = e^x e^t$

$\Rightarrow \int_{x_0}^{x} e^{-u}\,du = \int_0^t e^s\,ds$

$\Rightarrow e^{-x_0} - e^{-x} = e^t - 1$

$\Rightarrow e^{-x} = 1 - e^t + e^{-x_0}$

$\Rightarrow x = \ln\left(\frac{1}{1-e^t+e^{-x_0}}\right)$

(b) $\frac{dx}{dt} = x(t+1) + t + 1 = (x+1)(t+1)$

$\Rightarrow \int_{x_0}^{x} \frac{1}{u+1}\,du = \int_0^t (s+1)\,ds$

$\Rightarrow \ln\left(\frac{x+1}{x_0+1}\right) = \frac{t^2}{2} + t$

$\Rightarrow x+1 = (x_0+1)e^{t^2/2+t}$

$\Rightarrow x = (x_0+1)e^{t^2/2+t} - 1$

(c) $\frac{dx}{dt} = \frac{t^2+2}{x}$

$\Rightarrow \int_{x_0}^{x} u\,du = \int_0^t (s^2+2)\,ds$

$\Rightarrow \frac{x^2}{2} - \frac{x_0^2}{2} = \frac{t^3}{3} + 2t$

$\Rightarrow x^2 = x_0^2 + \frac{2}{3}t^3 + 4t$

$$\Rightarrow x = \begin{cases} \sqrt{x_0^2+\frac{2}{3}t^3+4t}, & x_0>0 \\ -\sqrt{x_0^2+\frac{2}{3}t^3+4t}, & x_0<0 \end{cases}$$

#2

Consider the following initial value problem

$$\begin{cases} \frac{dy}{dt} - a(t)y = 0, \\ y(0) = y_0 \end{cases}$$

where $a(t)$ is a smooth real valued function and $y_0 \in \mathbb{R}$.

(a) Show that the solution $y(t)$ is positive for all t, negative for all t, or zero for all t.

(b) If $y_0 = 1$, find a function $a(t)$ such that the solution $y(t)$ has $y(1) = 2$.

Solution:

(a) Since $y=0$ is a solution, it follows that any solution $y(t) \neq 0$ cannot change sign since this would violate existence and uniqueness, i.e., solutions cannot intersect.

(b) Let $a(t) = a$, where $a \in \mathbb{R}$ is a constant. Therefore,

$$y(t) = e^{at}$$

$$\Rightarrow 2 = e^{a}$$

$$\Rightarrow a = \ln(2).$$

Therefore,

$$y(t) = e^{\ln(2)t}, \quad a(t) = \ln(2).$$

#3

Assuming $y(0)=y_0$, find the solution to the following ordinary differential equation by making the indicated change of variables.

(a) $\frac{dy}{dt}=e^{t-y}-e^{t}$, $(v=e^{y})$

Solution:

(a) If $v=e^{y}$ then,

$$\frac{dv}{dt}=e^{y}\frac{dy}{dt}$$

$$=ve^{t}\left(\frac{1}{v}-1\right)$$

$$=e^{t}(1-v)$$

$$\Rightarrow \int_{e^{y_0}}^{e^{y}} \frac{1}{1-v}\,dv=\int_{0}^{t} e^{s}ds$$

$$\Rightarrow \ln\left(\frac{1-e^{y}}{1-e^{y_0}}\right)=e^{t}-1$$

$$\Rightarrow 1-e^{y}=(1-e^{y_0})e^{e^{t}-1}$$

$$\Rightarrow e^{y}=(e^{y_0}-1)e^{e^{t}-1}+1$$

$$\Rightarrow y=\ln\left[(e^{y_0}-1)e^{e^{t}-1}+1\right]$$

$\blacksquare$

#5

Consider the following initial value problem

$$\frac{dx}{dt} = -x^{1/3}$$

$$x(0)=0$$

(a) For $t_0<0$, show that the function

$$x(t)=\begin{cases}(2/3)^{3/2}(t_0-t)^{3/2}, & t<t_0,\\ 0, & t\geq t_0\end{cases}$$

is a differentiable function with a continuous derivative.

(b) Show that for $t_0<0$ the function defined in part (a) is a solution.

(c) Sketch $x(t)$ for $t_0=-1,-2,-3$.

Solution:

(a) Differentiating, we have that

$$\frac{dx}{dt} = \begin{cases}-(2/3)^{1/2}(t_0-t)^{1/2}, & t<t_0\\ 0, & t\geq t_0\end{cases}$$

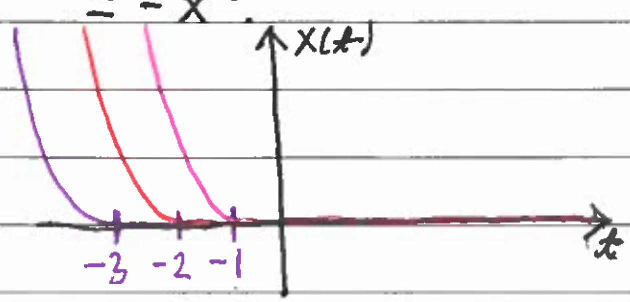
Since $\lim_{t\to t_0^-}\frac{dx}{dt}=\lim_{t\to t_0^+}\frac{dx}{dt}=0$ it follows that $x(t)$ is differentiable with continuous derivative.

(b) $$\frac{dx}{dt}=\begin{cases}-(2/3)^{1/2}(t_0-t)^{1/2}, & t<t_0\\ 0, & t\geq t_0\end{cases}$$

$$=\begin{cases}-\left[(2/3)^{3/2}\right]^{1/3}\left[(t_0-t)^{3/2}\right]^{1/3}, & t<t_0\\ 0 & t\geq t_0\end{cases}$$

$$=-x^{1/3}.$$

(c)



(d) The existence and uniqueness of solutions is not violated since $f(x)=-x^{1/3}$ is not differentiable at $x=0$.

#6

For the following diffential equations, sketch the solution curves $x(t)$ for different initial conditions.

(a) $\frac{dx}{dt} = 4x^2 - 16$

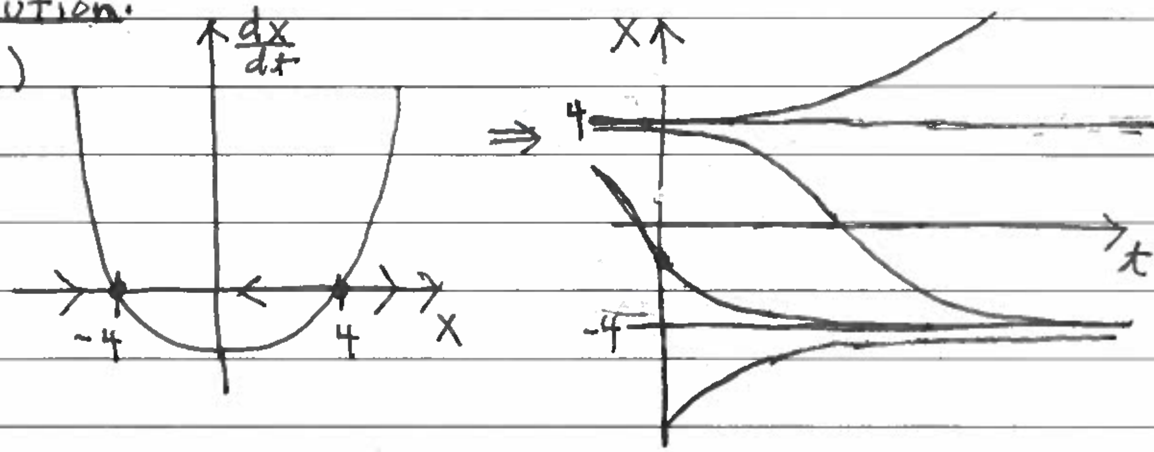
(b) $\frac{dx}{dt} = 1 - x^{14}$

(c) $\frac{dx}{dt} = x - x^3$

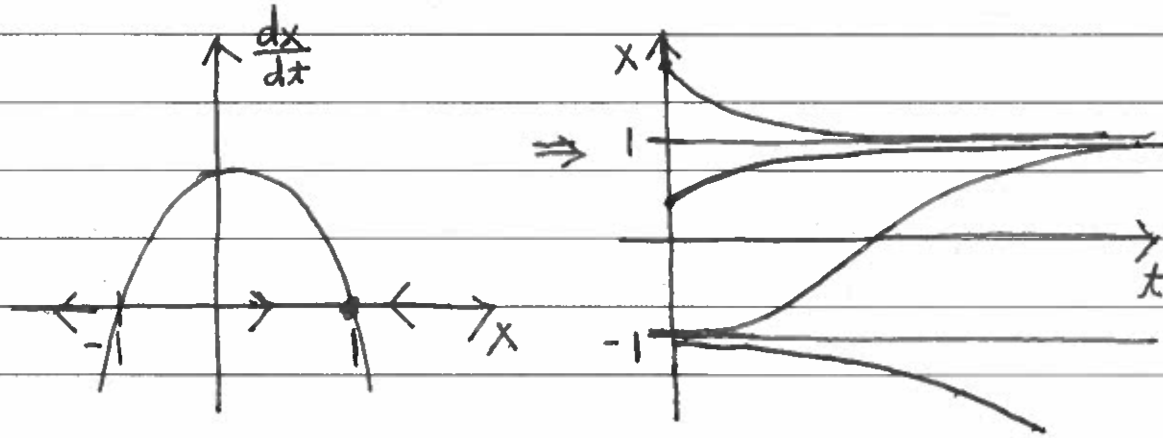
(d) $\frac{dx}{dt} = e^{-x}\sin(x)$

Solution:

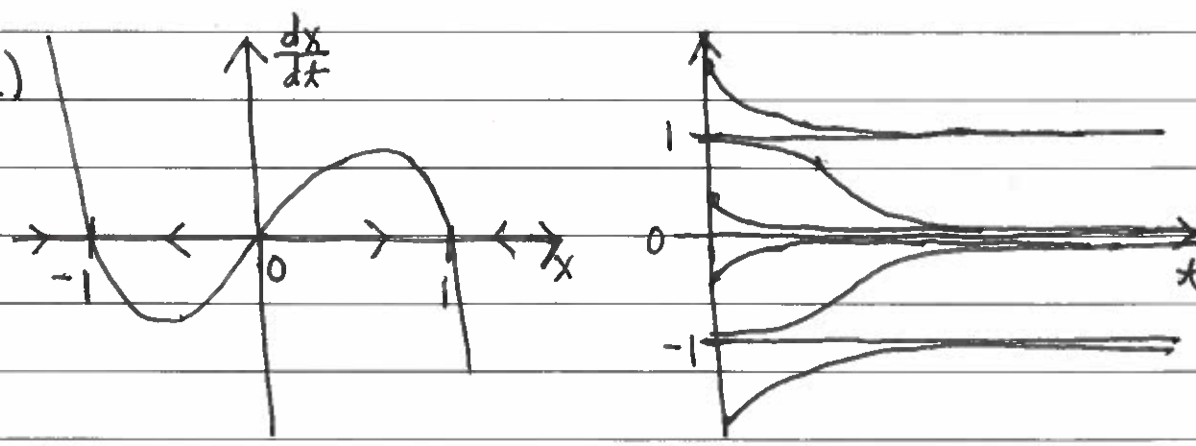
(a)



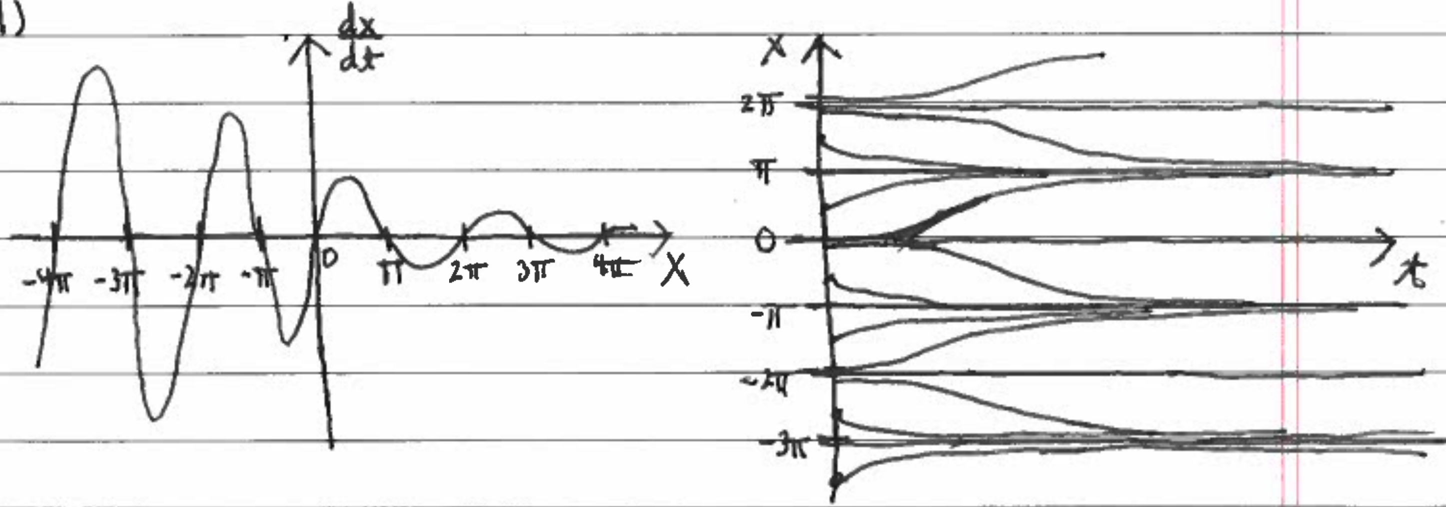
(b)



(c)

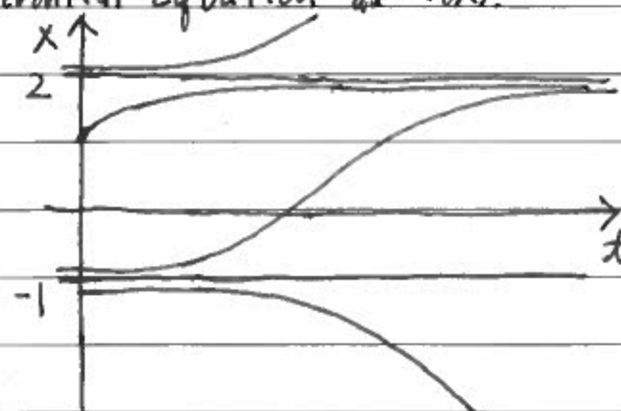


(d)



#7

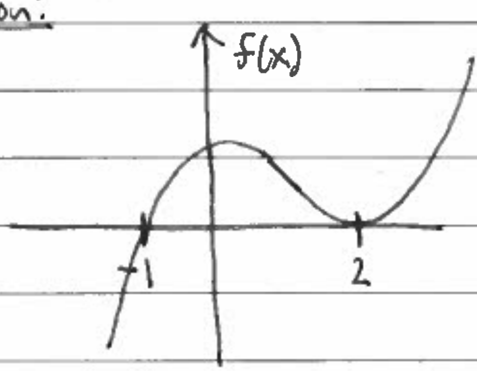
The curves $x(t)$ illustrated below correspond to solution curves for the differential equation $\frac{dx}{dt} = f(x)$.



(a) Sketch a graph of $f(x)$ that is consistent with this figure.
(b) Give a formula for $f(x)$ that is consistent with this figure.

Solution:

(a)



(b) $\frac{dx}{dt} = (x-2)^2(x+1)$

#8

For each of parts (a)-(e), find an equation $\frac{dx}{dt} = f(x)$ with the stated properties, or if there are no examples, explain why not.

(a) Every real number is a fixed point

(b) Every integer is a fixed point, and there are no others.

(c) There are precisely three fixed points and no others.

(d) There are no fixed points.

(e) There are precisely 100 fixed points.

Solution:

(a) $\frac{dx}{dt} = 0$

(b) $\frac{dx}{dt} = \sin(n\pi x)$

(c) $\frac{dx}{dt} = (x-1)(x-2)(x-3)$

(d) $\frac{dx}{dt} = 1$

(e) $\frac{dx}{dt} = (x-1)(x-2)\cdots(x-100)$

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