MTH 352/652
Homework #2

Due Date: January 31, 2025

2 1. Find all real valued solutions to the two-dimensional Laplace equation $u_{xx} + u_{yy} = 0$ that depend on the radial coordinate $r = \sqrt{x^2 + y^2}$.

1 2. Suppose $u(x,t)$ and $v(x,t)$ are infinitely differentiable functions in both x and t that satisfy the following system of equations:

$$u_t = v_x,$$
$$v_t = u_x.$$

(a) Show that both u and v are solutions to the wave equation $u_{tt} = u_{xx}$, $v_{tt} = v_{xx}$. Which result from multivariable calculus do you need to justify the conclusion?

(b) Conversely, given a solution $u(x,t)$ to the wave equation, can you construct a function $v(x,t)$ such that $u(x,t)$, $v(x,t)$ solve $u_t = v_x$ and $v_t = u_x$.

1 3. Classify the following differential equations as either (i) homogenous linear, (ii) inhomogeneous linear, or (iii) nonlinear.

(a) $u_t = x^2 u_{xx} + 2x u_x$

(b) $-u_{xx} - u_{yy} = \cos(u)$

(c) $u_{xx} + 2y u_{yy} = 3$

(d) $u_t + u u_x = 3u$

(e) $e^y u_x = e^x u_y$

(f) $u_t = 5u_{xxx} + x^2 u + x$

1 4. Suppose L and M are linear operators. Prove that the following are also linear operators:

(a) $L - M$.

(b) $3L$.

(c) fL, where f is an arbitrary function.

(d) $L \circ M$.

5. The displacement $u(x,t)$ of a forced violin string is modeled by the PDE $u_{tt} = 4u_{xx} + F(x,t)$. When $F(x,t) = \cos(x)$, the solution is $u(x,t) = \cos(x - 2t) + \frac{1}{4}\cos(x)$, while when $F(x,t) = \sin(x)$, the solution is $u(x,t) = \sin(x-2t) + \frac{1}{4}\sin(x)$. Find a solution when the forcing function $F(x,t)$ is

(a) $\cos(x) - 5\sin(x)$,

(b) $\sin(x - 3)$.

1 6. Find the general solution to the following partial differential equations

(a) $u_x = 0$

(b) $u_t = 1$

(c) $u_t = x - t$

(d) $u_t + 3u = 0$

(e) $u_x + tu = 0$

(f) $u_{tt} + 4u = 0$

2 7. Solve the following initial value problems and graph the solutions at times $t = 0, 1, 2, 3$.

(a) $u_t - 3u_x = 0$, $u(x, 0) = e^{-x^2}$

(b) $u_t + 2u_x = 0$, $u(x, -1) = x/(1 + x^2)$

(c) $u_t + u_x + \frac{1}{2}u = 0$, $u(x, 0) = \tan^{-1}(x)$

(d) $u_t - 4u_x + u = 0$, $u(x, 0) = 1/(1 + x^2)$

1 8. Let $c \neq 0$. Prove that if the initial data satisifes $u(x, 0) = v(x) \to 0$ as $x \to \pm\infty$, then, for each fixed x, the solution to the advection equation $u_t + cu_x = 0$ satisfies $u(x, t) \to 0$ as $t \to \infty$.

2 9. Solve the following initial value problem $u_t + 2u_x = \sin(x)$, $u(x, 0) = \sin(x)$.

10. Consider the partial differential equation $u_t + u_x + u^2 = 0$, $u(x, 0) = f(x)$ for $x \in \mathbb{R}$ and $t \geq 0$.

(a) Find the general formula for the solution to this PDE.

(b) Show that if $f(x)$ is positive and bounded, i.e., $0 \leq f(x) \leq M$, then the solution exists for all $t > 0$, and $u(x, t) \to 0$ as $t \to \infty$.

(c) On the other hand, if $f(x)$ is negative somewhere, then show that the solution blows up in finite time: $\lim_{t \to \tau^-} u(y, t) = -\infty$ for some $\tau > 0$ and some $y \in \mathbb{R}$.

(d) Find a formula for the earliest blow-up time $\tau^* > 0$.

11. Consider the initial value problem $u_t - xu_x = 0$, $u(x, 0) = (x^2 + 1)^{-1}$.

(a) Write down the differential equation satisfied by the characteristic curves and sketch the curves. Note, you don't have to find an explicit formula for the characteristic curves.

(b) Sketch the solution $u(x, t)$ at times $t = 0, 1, 5, 10$.

(c) Compute $\lim_{t \to \infty} u(x, t)$.

12. Consider the initial value problem $u_t + (1 + x^2)u_x = 0$, $u(0, x) = f(x)$.

(a) Write down the differential equation satisfied by the characteristic curves and find an explicit solution for the characteristic curves.

(b) Sketch the characteristic curves.

(c) Write down the general solution $u(x, t)$.

(d) Discuss properties of your solution as t increases.

Homework #2

#1.

Find all real valued solutions to the two-dimensional Laplace equation $u_{xx}+u_{yy}=0$ that depend on the radial coordinate $r=\sqrt{x^2+y^2}$.

Solution:

If we assume $u(x,y)=f(q)$, where $q=r^2=x^2+y^2$, we have that

$$\frac{\partial u}{\partial x}=\frac{df}{dq}\frac{\partial q}{\partial x}=2x\frac{df}{dq}$$

$$\frac{\partial^2 u}{\partial x^2}=2\frac{df}{dq}+4x^2\frac{d^2f}{dq^2}$$

$$\frac{\partial^2 u}{\partial y^2}=2\frac{df}{dq}+4y^2\frac{d^2f}{dq^2}$$

Therefore,

$$4q\frac{d^2f}{dq^2}+4\frac{df}{dq}=0$$

$$\Rightarrow q\frac{d^2f}{dq^2}+\frac{df}{dq}=0$$

$$\Rightarrow \frac{d}{dq}\left(q\frac{df}{dq}\right)=0$$

$$\Rightarrow q\frac{df}{dq}=A,$$

where A is a constant. Consequently,

$$\frac{df}{dq}=\frac{A}{q}$$

$$\Rightarrow f=A\ln(q)+B,$$

where B is a constant. Setting $q=r^2$ it follows

$$f = A\ln(r^2) + B$$
$$= 2A\ln(r) + B$$
$$= C\ln(r) + B,$$

where C is a constant.

$\blacksquare$

#2.

Suppose $u(x,t)$ and $v(x,t)$ are infinitely differentiable functions in both x and t that satisfy the following equations:

$$u_t = v_x$$
$$v_t = u_x$$

(a) Show that both u and v are solutions to the wave equation

$$u_{tt} = u_{xx}, \quad v_{tt} = v_{xx}$$

(b) Conversely, given a solution $u(x,t)$ to the wave equation, can you construct a function $v(x,t)$ such that $u(x,t)$, $v(x,t)$ solve $u_t = v_x$ and $v_t = u_x$.

Solution:

(a) By Clairut's theorem

$$u_{tt} = v_{xt} = v_{tx} = u_{xx}$$
$$v_{tt} = u_{xt} = u_{tx} = v_{xx}$$

(b) Now, let $v = \int u_t\,dx$ denote the antiderivative of u_t with respect to the x variable in which the arbitrary function of t is set to 0. Therefore,

$$v_t = \int u_{tt}\,dx = \int u_{xx}\,dx = u_x$$
$$\Rightarrow v_{tt} = u_{xt}.$$

We also have that

$$v_x = u_t$$
$$\Rightarrow v_{xx} = u_{tx}.$$

By Clairut's theorem, $v_{tt} = v_{xx}$. $\blacksquare$

#3

Classify the following differential equations as either

(i) homogeneous linear

(ii) inhomogeneous linear

(iii) nonlinear

(a) $u_t = x^2 u_{xx} + 2x u_x$

(b) $-u_{xx} - u_{yy} = \cos(u)$

(c) $u_{xx} + 2y\, u_{yy} = 3$

(d) $u_t + u u_x = 3u$

(e) $e^y u_x = e^x u_y$

(f) $u_t = 5u_{xxx} + x^2 u + x$

Solution:

(a) i, (b) iii, (c) ii, (d) iii, (e) i, (f) ii

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#4

Suppose L and M are linear operators. Prove that the following are also linear operators:

(a) $L - M$

(b) $3L$

(c) fL

(d) $L \circ M$

Solution:

(a) $(L-M)[u+\lambda v] = L[u+\lambda v] - M[u+\lambda v] = L[u] + \lambda L[v] - M[u] - \lambda M[v]$

$\Rightarrow (L-M)[u+\lambda v] = L[u] - L[M] + \lambda(L[v] - M[v]) = (L-M)[u] + \lambda(L-M)[v]$

(b) $(3L)[u+\lambda v] = 3(L[u] + \lambda L[v]) = (3L)[u] + \lambda(3L)[v]$.

(c) $(fL)[u+\lambda v] = f(L[u] + \lambda L[v]) = (fL)[u] + \lambda(fL)[v]$

(d) $L \circ M[u+\lambda v] = L(M[u+\lambda v]) = L(M[u] + \lambda M[v]) = LM[u] + \lambda LM[v]$

$\Rightarrow L \circ M[u+\lambda v] = L \circ M[u] + \lambda L \circ M[v]$

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#6

Find the general solution to the following partial differential equations.

(a) $u_x = 0$

(b) $u_t = 1$

(c) $u_t = x - t$

(d) $u_t + 3u = 0$

(e) $u_x + tu = 0$

(f) $u_{tt} + 4u = 0$.

Solution:

(a) $u = f(t)$

(b) $u = t + f(x)$

(c) $u = xt - \frac{1}{2}t^2 + f(x)$

(d) $u = f(x)e^{-3t}$

(e) $u = f(t)e^{-tx}$

(f) $u = f(x)\cos(2t) + g(x)\sin(2t)$

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#7

Solve the following initial value problems and graph the solutions at times $t = 0, 1, 2, 3$.

(a) $u_t - 3u_x = 0$, $u(x,0) = e^{-x^2}$

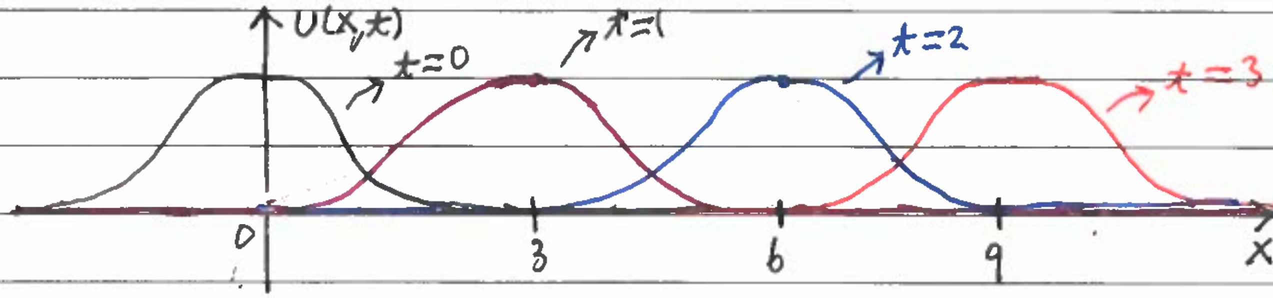
(b) $u_t + 2u_x = 0$, $u(x,-1) = x/1+x^2$

(c) $u_t + u_x + \frac{1}{2}u = 0$, $u(x,0) = \tan^{-1}(x)$

(d) $u_t - 4u_x + u = 0$, $u(x,0) = 1/1+x^2$

Solution:

(a) $u(x,t) = u_0(x+3t) = e^{-(x+3t)^2}$



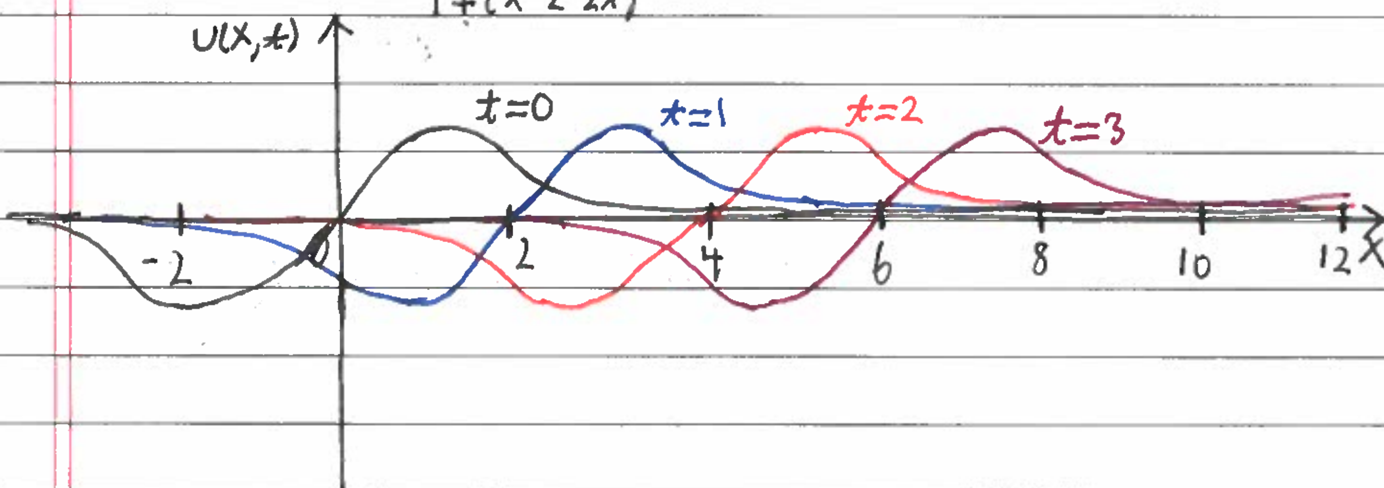
(b) $u(x,t)=f(x-2t)$

$\Rightarrow u(x,-1)=f(x+2)=x/1+x^2$

$$\Rightarrow f(x)=\frac{x-2}{1+(x-2)^2}$$

Therefore,

$$u(x,t)=\frac{x-2-2t}{1+(x-2-2t)^2}$$



(c) Letting $z=x-t$, $\tau=t$ we have

$$\frac{\partial}{\partial x}=\frac{\partial z}{\partial x}\frac{\partial}{\partial z}+\frac{\partial \tau}{\partial x}\frac{\partial}{\partial \tau}=\frac{\partial}{\partial z}$$

$$\frac{\partial}{\partial t}=\frac{\partial z}{\partial t}\frac{\partial}{\partial z}+\frac{\partial \tau}{\partial t}\frac{\partial}{\partial \tau}=-\frac{\partial}{\partial z}+\frac{\partial}{\partial \tau}$$

$$\Rightarrow \frac{\partial u}{\partial \tau}=-\frac{1}{2}u$$

$\Rightarrow u(z,\tau)=f(z)e^{-\frac{1}{2}\tau}$

$\Rightarrow u(x,t)=f(x-t)e^{-\frac{1}{2}t}$.

Applying initial conditions

$$u(x,0) = f(x) = \tan^{-1}(x).$$

Therefore,

$$u(x,t) = \tan^{-1}(x-t)e^{-\frac{1}{2}t}$$

(d) Letting $z = x+4t$, $\tau = t$ we have

$$\frac{\partial u}{\partial \tau} = -u$$

$$\Rightarrow u(z,\tau) = f(z)e^{-t}$$

$$\Rightarrow u(x,t) = f(x+4t)e^{-t}$$

From initial conditions we have

$$u(x,t) = \frac{e^{-t}}{1+(x+4t)^2}$$

$\blacksquare$

#8

Let $c \neq 0$. Prove that if the initial data satisfies $V(x) \to 0$ as $x \to \pm\infty$, then, for each fixed x, the solution to the advection equation satisfies $u(x,t) \to 0$ as $t \to \infty$.

Solution:

We know the solution is given by

$$u(x,t) = V(x-ct)$$

$$\Rightarrow \lim_{t\to\infty} u(x,t) = \lim_{t\to\infty} V(x-ct) = V\left(\lim_{t\to\infty}(x-ct)\right) = 0.$$

$\blacksquare$

#9.

Solve the following initial value problem :

$$u_t + 2u_x = \sin(x)$$
$$u(x,0) = \sin(x)$$

Solution:

Letting $z = x - 2t$, $\tau = t$, we have that

$$\frac{\partial u}{\partial \tau} = \sin(z + 2\tau)$$

$$\Rightarrow u = -\frac{1}{2}\cos(z + 2\tau) + f(z)$$
$$\Rightarrow u(x,t) = -\frac{1}{2}\cos(x) + f(x - 2t)$$

Now,

$$u(x,0) = \sin(x) = -\frac{1}{2}\cos(x) + f(x)$$
$$\Rightarrow f(x) = \sin(x) + \frac{1}{2}\cos(x).$$

Therefore,

$$u(x,t) = -\frac{1}{2}\cos(x) + \sin(x-2t) + \frac{1}{2}\cos(x-2t)$$

$\blacksquare$