MTH 352/652
Homework #3

Due Date: February 07, 2025

2 1. Consider the initial value problem $u_t - xu_x = 0$, $u(x,0) = (x^2+1)^{-1}$.

   (a) Write down the differential equation satisfied by the characteristic curves and sketch the curves. Note, you don't have to find an explicit formula for the characteristic curves.

   (b) Sketch the solution $u(x,t)$ at times $t = 0, 1, 5, 10$.

   (c) Compute $\lim_{t\to\infty} u(x,t)$.

2. Consider the initial value problem $u_t + (1+x^2)u_x = 0$, $u(0,x) = f(x)$.

   (a) Write down the differential equation satisfied by the characteristic curves and find an explicit solution for the characteristic curves.

   (b) Sketch the characteristic curves.

   (c) Write down the general solution $u(x,t)$.

   (d) Discuss properties of your solution as t increases.

3. Consider the advection equation with a speed that varies in time and space:

$$u_t + c(x,t)u_x = 0$$
$$u(x,0) = u_0(x),$$

   where $x \in \mathbb{R}$ and $t \in \mathbb{R}^+$. The characteristic curves are solutions to the following differential equation:

$$\frac{dx}{dt} = c(x,t).$$

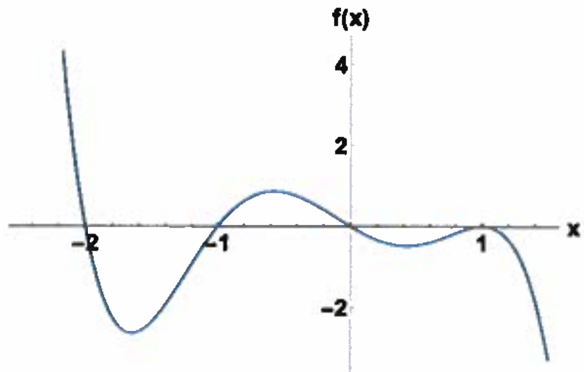
   (a) Show that any solution to the above advection equation is constant along the characteristic curves.

   (b) Suppose that the general form of the solutions to the characteristic equation can be written in the form $g(x(t),t) = k$, where k is an arbitrary constant. Show that $u(x,t) = f(g(x,t))$ is a solution to the above advection equation for any smooth function f.

2 4. Consider the following advection equation

$$u_t + f(x)u_x = 0,$$
$$u(x,0) = \frac{1}{1+x^2},$$

where $f(x)$ is plotted below:



(a) Sketch a graph of $u(x,0)$ as a function of x.

(b) Sketch a plot of the characteristic curves as a function of time t. Be sure to include enough of the curves to fully illustrate all possible types of curves.

(c) Compute

$$u^*(x) = \lim_{t\to\infty} u(t,x)$$

and sketch a graph of $u^*(x)$.

2 5. Show that the advection-diffusion-decay equation

$$u_t = Du_{xx} - cu_x - \lambda u$$

can be transformed into the diffusion equation $w_t = Dw_{xx}$ by the transformation

$$u(x,t) = w(x,t)e^{\alpha x - \beta t},$$

for some values of α and β.

6. If $z = x + iy$, where $x, y \in \mathbb{R}$, find $\text{Re}(1/z)$ and $\text{Im}(1/z)$.

7. If z is complex number, show that

$$\text{Re}(iz) = -\text{Im}(z) \text{ and } \text{Im}(iz) = \text{Re}(z).$$

1/2 8. Show that $z\overline{z} = |z|^2$ for any complex number z.

9. Show that $|z+w|^2 - |z-w|^2 = 4\text{Re}(z\overline{w})$ for any complex numbers z, w.

1/2 10. For $\theta \in \mathbb{R}$ prove that

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} \text{ and } \sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

$\frac{1}{2}$ 11. For $n \in \mathbb{Z}$, prove the following:

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta).$$

$\frac{1}{2}$ 12. Show that if $m, n \in \mathbb{Z}$ then

$$\int_0^{2\pi} e^{in\theta} e^{-im\theta} d\theta = \begin{cases} 2\pi & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}.$$

13. Calculate the Fourier transform of the "triangle function":

$$f(x) = \begin{cases} x+1 & \text{if } -1 \leq x \leq 0 \\ 1-x & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x < -1 \text{ or } x > 1 \end{cases}.$$

14. Find the Fourier transform of the function u defined by $u(x) = e^{-ax}$ if $x > 0$, and $u(x) = 0$ if $x \leq 0$.

15. Compute $\mathcal{F}[xe^{-ax^2}](k)$, where $a > 0$ is a constant.

2 16. Given that $\mathcal{F}[xe^{-|x|}](k) = \dfrac{4ik}{(1+k^2)^2}$, find $\mathcal{F}\left[\dfrac{x}{(1+x^2)^2}\right](k)$.

Homework #3

#1

Consider the initial value problem $u_t - xu_x = 0$, $u(x,0) = (x^2+1)^{-1}$.

(a) Write down the differential equation satisfied by the characteristic curves.

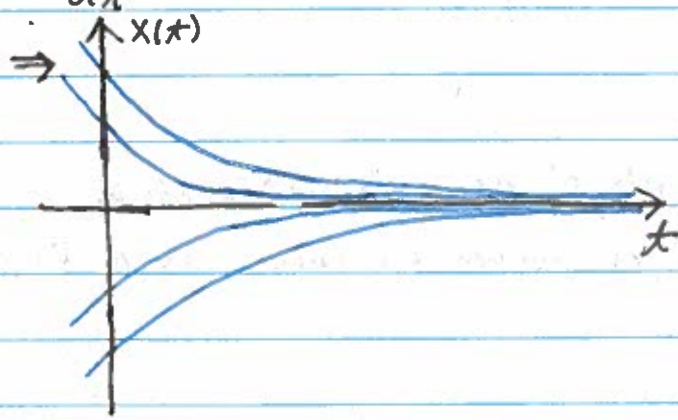
(b) Sketch the solution $u(x,t)$ at times $t = 0, 1, 5, 10$.

(c) Compute

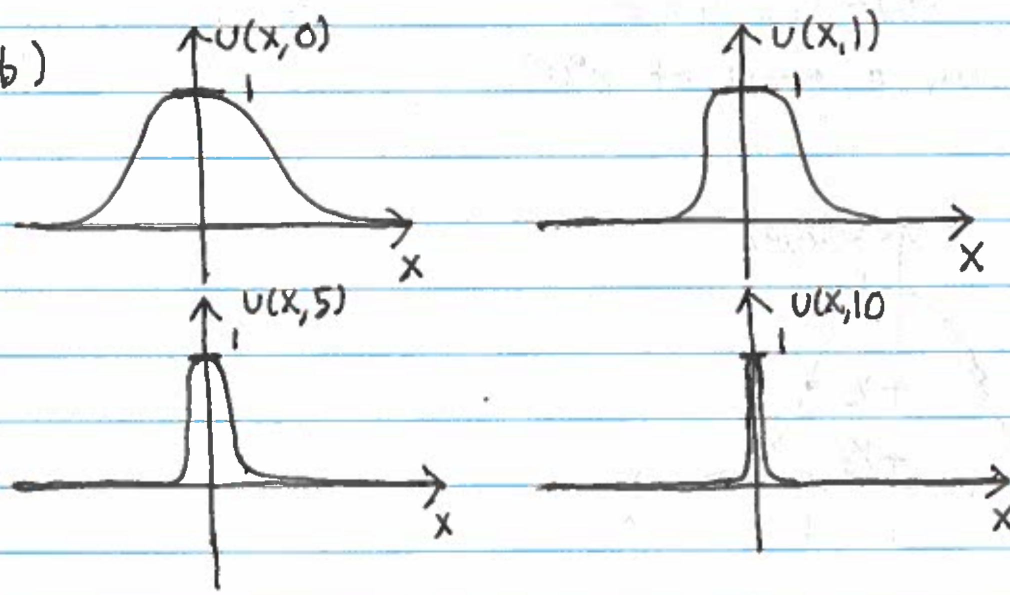
$$\lim_{t\to\infty} u(t,x).$$

Solution:

(a) $\dfrac{dx}{dt} = -x$



(b)



(c) $\lim_{t\to\infty} u(t,x) = \begin{cases} 0, & x \neq 0 \\ 1, & x = 0 \end{cases}$

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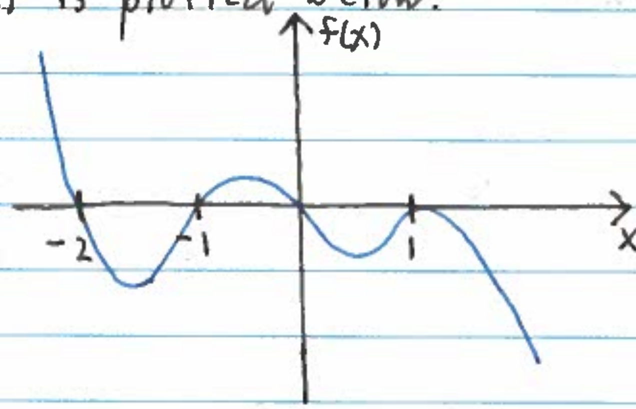
#4

Consider the following advection equation

$$u_t + f(x)u_x = 0$$
$$u(x,0) = \frac{1}{1+x^2}$$

Where $f(x)$ is plotted below.



(a) Sketch a graph of $u(x,0)$ as a function of x.

(b) Sketch a plot of the characteristic curves as a function of time t.

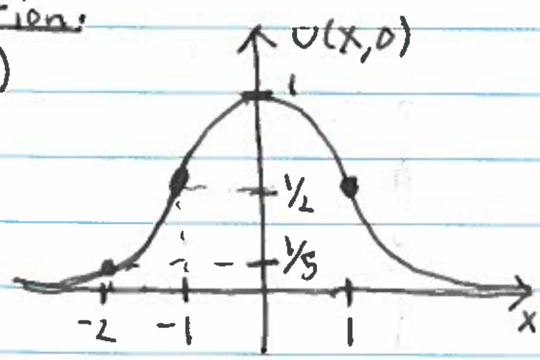
(c) Compute

$$u^*(x) = \lim_{t\to\infty} u(x,t)$$

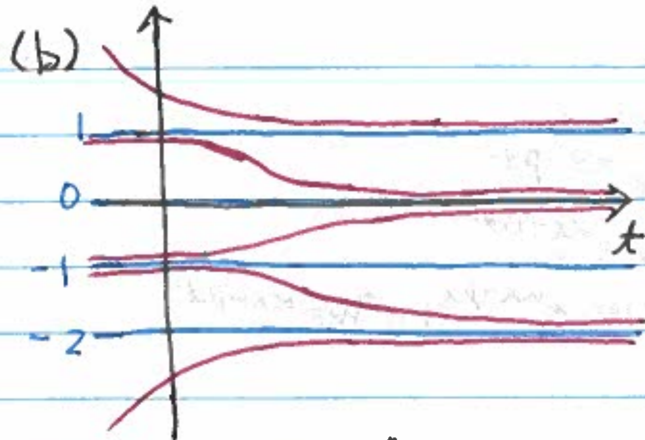
and sketch a graph of $u^*(x)$.

Solution:

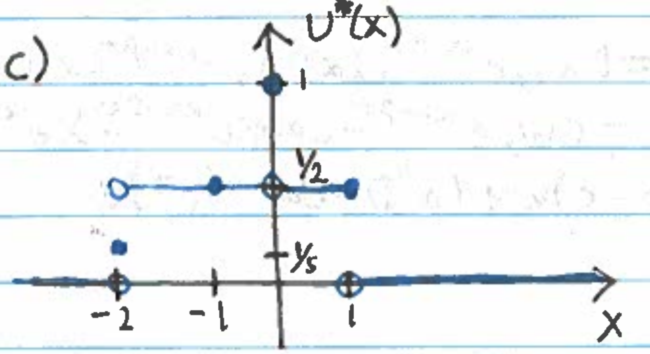
(a)



(b)



(c)



$$\lim_{t\to\infty} u(x,t) = \begin{cases} 0, & x<-2,\ x>1 \\ 1/2, & -2<x<0,\ 0<x\leq 1 \\ 1/5, & x=-2 \\ 1, & x=0 \end{cases}$$

#5

Show that the advection-diffusion-decay equation

$$u_t = Du_{xx} - cu_x - \lambda u$$

can be transformed into the diffusion equation

$$w_t = Dw_{xx}$$

by the transformation

$$u(x,t) = w(x,t)e^{\alpha x - \beta t}$$

for some values of α and β.

Solution:

Assume $u = we^{\alpha x - \beta t}$. Therefore,

$$U_t = W_t e^{\alpha x-\beta t} - \beta w e^{\alpha x - \beta t}$$
$$U_x = W_x e^{\alpha x-\beta t} + \alpha w e^{\alpha x - \beta t}$$
$$U_{xx} = W_{xx} e^{\alpha x-\beta t} + 2\alpha W_x e^{\alpha x - \beta t} + \alpha^2 w e^{\alpha x - \beta t}.$$

Therefore,

$$U_t = DU_{xx} - cU_x - \lambda U$$
$$\Rightarrow W_t e^{\alpha x-\beta t} - \beta w e^{\alpha x-\beta t} = Dw_{xx} e^{\alpha x-\beta t} + 2\alpha D w_x e^{\alpha x-\beta t} + \alpha^2 D w e^{\alpha x-\beta t} - cW_x e^{\alpha x-\beta t} - c\alpha w e^{\alpha x-\beta t} - \lambda e^{\alpha x-\beta t}$$
$$\Rightarrow W_t = DW_{xx} + (2\alpha D - c)w_x + (\alpha^2 D - c\alpha - \lambda + \beta)w$$

Consequently,

$$\alpha = c/2D$$
$$\Rightarrow \frac{c^2}{4D} - \frac{c^2}{2D} - \lambda + \beta = 0$$
$$\Rightarrow \beta = \lambda - \frac{c^2}{4D}.$$

#8

Show that $z \cdot \bar{z} = |z|^2$ for any complex number z.

Solution:

Let $z = |z|e^{i\theta}$. Therefore, $z \cdot \bar{z} = |z|e^{i\theta} \cdot |z|e^{-i\theta} = |z|^2$.

#10

For $\theta \in \mathbb{R}$ prove that

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}.$$

Solution:

$$\frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{\cos\theta + i\sin\theta + \cos\theta - i\sin\theta}{2} = \cos\theta.$$

$$\frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{\cos\theta + i\sin\theta - \cos\theta + i\sin\theta}{2i} = \sin\theta$$

$\blacksquare$

#11

For $n \in \mathbb{Z}$, prove the following

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

Solution:

$$\begin{aligned}(\cos\theta + i\sin\theta)^n &= (e^{i\theta})^n \\ &= e^{in\theta} \\ &= \cos(n\theta) + i\sin(n\theta)\end{aligned}$$

$\blacksquare$

#12

Show that if $m, n \in \mathbb{Z}$ then

$$\int_0^{2\pi} e^{in\theta} e^{-in\theta} = \begin{cases} 2\pi & \text{if } m=n \\ 0 & \text{if } m \neq n. \end{cases}$$

Solution:

1. If $m=n$, $\int_0^{2\pi} e^{in\theta} e^{-im\theta} d\theta = \int_0^{2\pi} 1\, d\theta = 2\pi$.

2. If $m \neq n$, $\int_0^{2\pi} e^{int} e^{-imt} dt = \int_0^{2\pi} e^{i(n-m)\theta} d\theta$

$$= \frac{1}{i(n-m)} e^{i(n-m)\theta} \Big|_0^{2\pi}$$

$$= \frac{1}{i(n-m)} \left(e^{i(n-m)2\pi} - 1\right)$$

$$= 0.$$

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#16

Given that

$$\mathcal{F}[xe^{-|x|}](k) = \frac{4ik}{(1+k^2)^2}$$

find $\mathcal{F}\left[\frac{x}{(1+x^2)^2}\right](k)$.

Solution:

$$\mathcal{F}\left[\frac{x}{(1+x^2)^2}\right](k) = \int_{-\infty}^{\infty} \frac{x}{(1+x^2)^2} e^{ikx} dx$$

$$= \int_{-\infty}^{\infty} \frac{-x}{(1+x^2)^2} e^{-ikx} dx$$

$$= \frac{1}{2\pi} \cdot \frac{i}{4} \int_{-\infty}^{\infty} \frac{4ix}{(1+x^2)^2} e^{-ikx} dx$$

$$= \frac{i}{4} k e^{-|k|}.$$

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