MTH 352/652
Homework #4

Due Date: February 14, 2025

In all of these problems you can assume that the relevant functions are Schwartz class functions.

1. Verify the following properties of the Fourier transform:
   (a) $\mathcal{F}[u(x)](k) = 2\pi\mathcal{F}^{-1}[u(x)](-k)$.
   (b) $\mathcal{F}[e^{iax}u(x)](k) = \hat{u}(k+a)$.
   (c) $\mathcal{F}[u(x+a)](k) = e^{-iak}\hat{u}(k)$.

2. If f and g are Schwartz class functions, show that
$$(f * g)(x) = (g * f)(x).$$

3. Let $u(x) = e^{-|x|}$.
   (a) Find $\mathcal{F}[u(x)](k)$.
   (b) Find $(u * u)(x)$.
   (c) Find the inverse Fourier transform of the function
$$f(k) = \frac{1}{(1+k^2)^2}.$$

4. Consider the following initial value problem for the heat equation with proportional heat loss:
$$u_t = Du_{xx} - au, \ x \in \mathbb{R}, \ t > 0,$$
$$u(x,0) = e^{-x^2},$$
   where $D > 0$ and $a > 0$ are constants.
   (a) Show that if u is a solution to this PDE then the energy $E(t) = \int_{-\infty}^{\infty} u^2 dx$ is decreasing in time.
   (b) Using this energy, prove that solutions to this PDE are unique.
   (c) Using Fourier transforms, find a formula for the solution to this initial value problem. **Hint:** I am not expecting you to use the convolution theorem to solve this problem. Since $u(x,0)$ is a Gaussian you can explicitly compute all of the Fourier and inverse Fourier transforms.
   (d) Explain what effect of the additional term $-au$ has on the behavior of solutions compared with the heat equation without loss.
   (e) Show that by changing variables $v(x,t) = e^{at}u(x,t)$ that v satisfies the heat equation $v_t = Dv_{xx}$.

5. Consider the following initial value problem for the heat equation with advection:

$$u_t = Du_{xx} - cu_x, \ x \in \mathbb{R}, \ t > 0,$$
$$u(x,0) = e^{-x^2},$$

where $D > 0$ and $c > 0$ are constants.

(a) Show that if u is a solution to this PDE then the energy $E(t) = \int_{-\infty}^{\infty} u^2 dx$ is decreasing in time.

(b) Using this energy, prove that solutions to this PDE are unique.

(c) Using Fourier transforms find a formula for the solution to this initial value problem. **Hint:** I am not expecting you to use the convolution theorem to solve this problem. Since $u(x,0)$ is a Gaussian you can explicitly compute all of the Fourier and inverse Fourier transforms.

(d) Assuming $D = 1$ and $c = 1$, on the same axis sketch $u(x,0)$, $u(x,1)$, and $u(x,2)$ as functions of x. Explain qualitatively the behavior of the solution as time increases.

(e) By changing variables to $\tau = t$ and $X = x - ct$ show that u satisfies $u_\tau = Du_{XX}$.

Homework #4

#1.

Verify the following properties of the Fourier transform:

(a) $\mathcal{F}[u](k) = 2\pi\,\mathcal{F}^{-1}[u](-k)$.

(b) $\mathcal{F}[e^{iax}u](k) = \hat{u}(k+a)$

(c) $\mathcal{F}[u(x+a)](k) = e^{-iak}\hat{u}(k)$

Solution:

(a)
$$\mathcal{F}[u] = \int_{-\infty}^{\infty} u(x)e^{ikx}dx$$
$$= \int_{-\infty}^{\infty} u(-y)e^{-iky}dy$$
$$= \frac{2\pi}{2\pi}\int_{-\infty}^{\infty} u(-y)e^{-iky}dy$$
$$= 2\pi\,\mathcal{F}^{-1}[u(-x)](-k).$$

(b)
$$\mathcal{F}[e^{iax}u(x)](k) = \int_{-\infty}^{\infty} u(x)e^{i(k+a)x}dx$$
$$= \mathcal{F}[u](k+a).$$

(c)
$$\mathcal{F}[u(x+a)](k) = \int_{-\infty}^{\infty} u(x+a)e^{ikx}dx$$
$$= \int_{-\infty}^{\infty} u(y)e^{iky}e^{-ika}dy$$
$$= e^{-ika}\int_{-\infty}^{\infty} u(y)e^{iky}dy$$
$$= e^{-ika}\mathcal{F}[u](k)$$

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#2

If f and g are Schwartz class functions, show that
$$(f*g)(x) = (g*f)(x)$$

Solution:
$$f*g = \mathcal{F}^{-1}[\mathcal{F}[f*g]] = \mathcal{F}^{-1}[\hat{f}\cdot\hat{g}] = \mathcal{F}^{-1}[\hat{g}\cdot\hat{f}] = \mathcal{F}^{-1}[g*f] = g*f.$$

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#3.

Let $u(x)=e^{-|x|}$

(a) Find $\mathcal{F}[u(x)](K)$

(b) Find $(u*u)(x)$.

(c) Find the inverse Fourier transform of the function

$$f(K)=\frac{1}{(1+K^2)^2}.$$

Solution:

(a) $$\mathcal{F}[u(x)](K)=\int_{-\infty}^{\infty}e^{-|x|}e^{ikx}dx$$
$$=\int_{-\infty}^{0}e^{x}e^{ikx}dx+\int_{0}^{\infty}e^{-x}e^{ikx}dx$$
$$=\frac{1}{1+iK}-\frac{1}{-1+iK}$$
$$=\frac{1}{1+iK}+\frac{1}{1-iK}$$
$$=\frac{2}{1+K^2}.$$

(b) $$(u*u)(x)=\int_{-\infty}^{\infty}e^{-|x-y|}e^{-|y|}dy$$
$$=\begin{cases}\int_{-\infty}^{x}e^{y-x}e^{y}dy+\int_{x}^{0}e^{x-y}e^{y}dy+\int_{0}^{\infty}e^{x-y}e^{-y}dy, & x<0\\ \int_{-\infty}^{0}e^{y-x}e^{y}dy+\int_{0}^{x}e^{y-x}e^{-y}dy+\int_{x}^{\infty}e^{x-y}e^{-y}dy, & x>0\end{cases}$$
$$=\begin{cases}\frac{1}{2}e^{-x}e^{2x}-xe^{x}+\frac{1}{2}e^{x}, & x<0\\ \frac{1}{2}e^{-x}+xe^{-x}+\frac{1}{2}e^{x}e^{-2x}, & x>0\end{cases}$$
$$=\begin{cases}e^{x}(1-x), & x<0\\ e^{-x}(1+x), & x>0\end{cases}$$
$$=(1+|x|)e^{-|x|}$$

(c) By the convolution theorem,

$$\mathcal{F}[u*u]=\mathcal{F}[u]\cdot\mathcal{F}[u]$$
$$=\frac{4}{(1+K^2)^2}$$

$$\Rightarrow \mathcal{F}^{-1}\left[\frac{4}{(1+K^2)^2}\right](x)=\frac{1}{4}u*u=\frac{(1+|x|)e^{-|x|}}{4}$$

#4.

Consider the following initial value problem for the heat equation with proportional heat loss:

$$u_t = Du_{xx} - au,\ x\in\mathbb{R},\ t>0,$$
$$u(x,0) = e^{-x^2},$$

where $D>0$ and $a>0$ are constants.

(a) Show that if u is a solution to this PDE then the energy $E(t)=\int_{-\infty}^{\infty} u^2\,dx$ is decreasing in time.

(b) Using this energy, prove that solutions to this PDE are unique.

(c) Using Fourier transforms, find a formula for the solution to this initial value problem.

(d) Explain what effect the additional term $-au$ has on the behavior of solutions when compared with the heat equation.

(e) Show that by changing variables $v(x,t)=e^{at}u(x,t)$ that v satisfies the heat equation $v_t = Dv_{xx}$.

Solution:

(a)
$$\begin{aligned}\frac{dE}{dt} &= \int_{-\infty}^{\infty} 2u_t u\,dx \\ &= \int_{-\infty}^{\infty} 2(Du_{xx}-au)u\,dx \\ &= 2Du_xu\Big|_{-\infty}^{\infty} - 2\int_{-\infty}^{\infty}(Du_x^2+au^2)\,dx \\ &= -2\left(\int_{-\infty}^{\infty} Du_x^2\,dx + au^2\right)dx \\ &\le 0.\end{aligned}$$

(b) ~~proof~~:

If u_1, u_2 solve the PDE and $v=u_1-u_2$ then v solves

$$v_t = Dv_{xx} - av,\ x\in\mathbb{R},\ t>0.$$
$$v(x,0)=0$$

Consequently, if $E(t)=\int_{-\infty}^{\infty} v^2 dx$ it follows that E is decreasing, $E\geq 0$, and $E(0)=0$. Therefore, for all t, $E(t)=0$ and thus $v=0$.

(c) Taking Fourier transforms of both sides we have

$$\hat{u}_t=-k^2 D\hat{u}-a\hat{u}$$
$$\hat{u}(k,0)=\sqrt{\pi}e^{-k^2/4}$$

Therefore,

$$\hat{u}(k,t)=\sqrt{\pi}e^{-k^2/4}e^{-(k^2D+a)t}$$
$$\Rightarrow u(x,t)=\sqrt{\pi}e^{-at}\mathcal{F}^{-1}\left[e^{-(1/4+Dt)\cdot k^2}\right](x)$$
$$=\sqrt{\pi}e^{-at}\mathcal{F}^{-1}\left[e^{-\frac{(1+4Dt)}{4}\cdot k^2}\right](x)$$
$$=e^{-at}\frac{\sqrt{\pi}}{2\pi}\mathcal{F}\left[e^{-\frac{(1+4Dt)}{4}k^2}\right](x)$$
$$=\frac{e^{-at}}{2}\cdot\sqrt{\frac{4}{1+4Dt}}\,e^{-x^2/1+4Dt}$$
$$=\frac{e^{-at}}{\sqrt{1+4Dt}}e^{-x^2/1+4Dt}$$

(d) The solution decays exponentially fast by the additional factor e^{-at}.

(e). Letting $v=e^{at}u$ we have that

$$v_t=ae^{at}u+e^{at}u_t=ae^{at}u+e^{at}(Du_{xx}-au)$$
$$\Rightarrow v_t=e^{at}u_{xx}=D\frac{\partial^2}{\partial x^2}(e^{at}u)=Dv_{xx}.$$

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#5

Consider the following initial value problem for the heat equation with advection:

$$u_t = Du_{xx} - cu_x,\ x \in \mathbb{R},\ t>0$$
$$u(x,0) = e^{-x^2}.$$

where $D>0$ and $c>0$ are constants.

(a) Show that if u is a solution to this PDE then the energy $E(t) = \int_{-\infty}^{\infty} u^2 dx$ is decreasing in time.

(b) Using this energy, prove that solutions are unique.

(c) Using Fourier transforms find a solution to this initial value problem.

(d) Assuming $D=1$ and $c=1$, on the same axis sketch $u(x,0)$, $u(x,1)$, $u(x,2)$ as functions of x. Explain qualitatively the behavior of the solution as time increases.

(e) By changing variables to $\tau = t$ and $X = x - ct$ show that u satisfies $u_\tau = Du_{XX}$.

Solution:

(a) $\frac{dE}{dt} = \int_{-\infty}^{\infty} 2uu_t\,dx = \int_{-\infty}^{\infty} 2u(Du_{xx} - cu_x)\,dx$

$$\Rightarrow \frac{dE}{dt} = 2uu_x\Big|_{-\infty}^{\infty} - 2\int_{-\infty}^{\infty} Du_x^2\,dx - c\int_{-\infty}^{\infty} \frac{d}{dx}u^2\,dx$$
$$= -2\int_{-\infty}^{\infty} Du_x^2\,dx - cu^2\Big|_{-\infty}^{\infty}$$
$$= -2\int_{-\infty}^{\infty} Du_x^2\,dx$$
$$\leq 0.$$

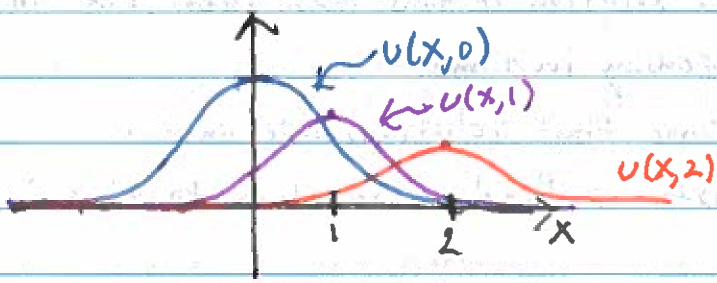
(b) See problem #4.

(c) Taking Fourier transform of both sides

$$\hat{u}_t = -k^2 D\hat{u} - ick\hat{u}$$
$$\hat{u}(k,0) = \sqrt{\pi}e^{-k^2/4}$$
$$\Rightarrow \hat{u}(k,t) = \sqrt{\pi}e^{-k^2/4}e^{-(k^2D + ick)t}$$

$$\Rightarrow u(x,t) = \sqrt{\pi}\,\mathcal{F}^{-1}\left[e^{-(1/4+Dt)k^2} e^{ickt}\right](x)$$
$$= \frac{\sqrt{\pi}}{2\pi}\,\mathcal{F}\left[e^{-(1/4+Dt)k^2} e^{-ickt}\right](x)$$
$$= \frac{\sqrt{\pi}}{2\pi}\,\mathcal{F}\left[e^{-(1/4+Dt)k^2}\right](x-ct)$$
$$= \frac{1}{\sqrt{1+4Dt}}\, e^{-(x-ct)^2/1+4Dt}$$

(d)



The solutions spread out like the heat equation, but also move to the right at sped c.

(e) Letting $\tau = t$ and $X = x - ct$ it follows that

$$\frac{\partial}{\partial t} = \frac{\partial \tau}{\partial t}\frac{\partial}{\partial \tau} + \frac{\partial X}{\partial t}\frac{\partial}{\partial X}$$
$$= \frac{\partial}{\partial \tau} - c\frac{\partial}{\partial X}$$
$$\frac{\partial}{\partial x} = \frac{\partial \tau}{\partial x}\frac{\partial}{\partial \tau} + \frac{\partial X}{\partial x}\frac{\partial}{\partial X}$$
$$= \frac{\partial}{\partial X}$$
$$\Rightarrow \frac{\partial^2}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial}{\partial x} = \frac{\partial}{\partial X}\frac{\partial}{\partial X} = \frac{\partial^2}{\partial X^2}.$$

Therefore,

$$u_t = D u_{xx} - c u_x$$
$$\Rightarrow u_\tau - c u_X = D u_{XX} - c u_X$$
$$\Rightarrow u_\tau = D u_{XX}.$$

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