MTH 352/652
Homework #5

Due Date: February 28, 2025

1. Consider the wave equation for $x \in \mathbb{R}$ and $t > 0$:

$$u_{tt} = c^2 u_{xx}$$

   (a) By taking the Fourier transform of this equation, show that if $u(x,t)$ is a solution then

$$\hat{u}(k,t) = F(k)e^{ikct} + G(k)e^{-ikct},$$

   for some functions F and G.

   (b) If $f(x) = \mathcal{F}^{-1}[F(k)]$ and $g(x) = \mathcal{F}^{-1}[G(k)]$, show that

$$u(x,t) = f(x-ct) + g(x+ct).$$

2. Solve the following PDE on the domain $x \in \mathbb{R}$ and $t \geq 0$:

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \\ u(x,0) &= \frac{1}{1+x^2}, \\ u_t(x,0) &= \sin(x). \end{aligned}$$

3. Let $\Psi(x)$ be the function defined by

$$\Psi(x) = \begin{cases} 1 & |x| < a \\ 0 & |x| \geq 0 \end{cases},$$

for some constant $a > 0$. Suppose $u(t,x)$ is a solution to the following PDE

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \\ u(x,0) &= 0, \\ u_t(x,0) &= \Psi(x). \end{aligned}$$

   (a) Sketch a graph of $\Psi(x)$.

   (b) Show that the solution to this PDE is given by

$$u(x,t) = \frac{1}{2c}|(x-ct, x+ct) \cap (-a,a)|,$$

   where here $|\cdot|$ denotes the length of an interval on the real line.

   (c) Sketch the solution at times $t = a/2c$, $t = a/c$, $t = 3a/2c$, $t = 2a/c$, and $t = 5a/c$.

4. In a long transmission line the voltage $V(x,t)$ and current $I(x,t)$ satisfy the transmission line equations:

$$I_x + CV_t + GV = 0 \text{ and } V_x + LI_t + RI = 0,$$

where $C > 0$ is the capacitance, $G > 0$ is the leakage, $R > 0$ is the resistance, and $L > 0$ is the inductance.

(a) Show that both V and I satisfy the telegraph equation

$$LCu_{tt} + (RC + LG)u_t + RGu = u_{xx}. \tag{1}$$

(b) If for the moment we assume that $R = G = 0$ show that Equation (1) become the wave equation. What is the speed? Why is this case unrealistic?

(c) If we let $c = /\sqrt{LC}$, $a = c^2(LG + RC)$ and $b = c^2RG$ show that Equation (1) becomes

$$u_{tt} + au_t + bu = c^2u_{xx}.$$

(d) By making a substitution of the form $u(x,t) = e^{-\lambda t}v(x,t)$ and by appropriately picking λ in terms of a, b, and c, show that Equation (4c) becomes

$$v_{tt} + kv = c^2v_{xx},$$

where k is a constant that depends on a, b, and c.

(e) Show that $k = 0$ only when $RC = LG$.

(f) In the case when $RC = LG$ with the initial conditions $v(x,0) = f(x)$ and $v_t(x,0) = 0$, show that a solution to the telegrapher's equation is given by

$$u(t,x) = \frac{e^{-\lambda t}}{2}\left(f(x-ct) + f(x+ct)\right).$$

What does this solution mean in practical terms? In particular, why is it useful for sending signals?

Remark This process of tuning the electrical parameters so the $RC = LG$ called Pupinizing the cable after one of its discoverers Michael Pupin.

5. Solve the following PDE for $x \in \mathbb{R}$ and $t \geq 0$:

$$u_{xx} - 3u_{xt} - 4u_{tt} = 0,$$
$$u(x,0) = e^{-x^2},$$
$$u_t(x,0) = e^x.$$

Hint: Factor the differential operator like we did in class.

Homework #5

#1

Consider the wave equation for $x \in \mathbb{R}$ and $t > 0$:

$$u_{tt} = c^2 u_{xx}$$

(a) By taking the Fourier transform of this equation, show that if $u(x,t)$ is a solution then

$$\hat{u}(k,t) = F(k)e^{ikct} + G(k)e^{-ikct},$$

for some functions F and G.

(b) If $f(x) = \mathcal{F}^{-1}[F(k)]$ and $g(x) = \mathcal{F}^{-1}[G(k)]$, show that

$$u(x,t) = f(x-ct) + g(x+ct).$$

Solution:

(a) Taking Fourier transforms we have

$$\hat{u}_{tt} = -k^2c^2\hat{u}$$

$$\Rightarrow \hat{u}(k,t) = F(k)e^{ikct} + G(k)e^{-ikct}$$

(b) Inverting, we have

$$\begin{aligned} u &= \mathcal{F}^{-1}[F(k)e^{ikct}] + \mathcal{F}^{-1}[G(k)e^{-ikct}] \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)e^{ikct}e^{-ikx}\,dk + \frac{1}{2\pi}\int_{-\infty}^{\infty} G(k)e^{-ikct}e^{-ikx}\,dk \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)e^{-ik(x-ct)}\,dk + \frac{1}{2\pi}\int_{-\infty}^{\infty} G(k)e^{-ik(x+ct)}\,dk \\ &= \mathcal{F}^{-1}[F(k)](x-ct) + \mathcal{F}^{-1}[G(k)](x+ct) \\ &= f(x-ct) + g(x+ct) \end{aligned}$$

$\blacksquare$

#2

Solve the following PDE on the domain $x \in \mathbb{R}$, $t \geq 0$:

$$u_{tt} = c^2 u_{xx},$$
$$u(x,0) = 1/(1+x^2),$$
$$u_t(x,0) = \sin(x).$$

Solution:

$$u(x,t) = \frac{1}{2}\frac{1}{1+(x-ct)^2} + \frac{1}{2}\frac{1}{1+(x+ct)^2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \sin(x)\,dx$$
$$= \frac{1}{2}\frac{1}{1+(x-ct)^2} + \frac{1}{2}\frac{1}{1+(x+ct)^2} + \frac{1}{2c}\left(\cos(x-ct) - \cos(x+ct)\right)$$

■

#3

Let $\psi(x)$ be the function defined by

$$\psi(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$$

for some constant $a > 0$. Suppose $u(x,t)$ is a solution to the following PDE

$$u_{tt} = c^2 u_{xx},$$
$$u(x,0) = 0,$$
$$u_t(x,0) = \psi(x).$$

(a) Sketch a graph of $\psi(x)$.

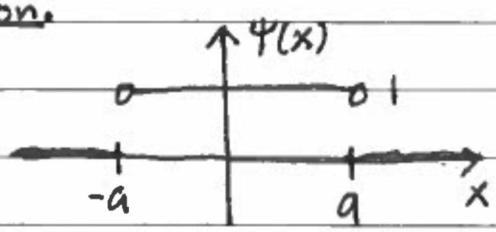
(b) Show that the solution to this PDE is given by

$$u(x,t) = \frac{1}{2c}\left|(x-ct, x+ct) \cap (-a,a)\right|.$$

(c) Sketch the solutions at times $t = a/2c$, a/c, $3a/2c$, $2a/c$, and $5a/c$.

Solution:

(a)



(b) Let

$$\chi(x,y,t)=\begin{cases}1, & y\in(x-ct,x+ct)\\ 0, & y\notin(x-ct,x+ct)\end{cases}$$

Therefore,

$$u(x,t)=\frac{1}{2c}\int_{-\infty}^{\infty}\chi(x,y,t)\,\psi(y)\,dy,$$

$$=\frac{1}{2c}\int_{A} 1\,dy,$$

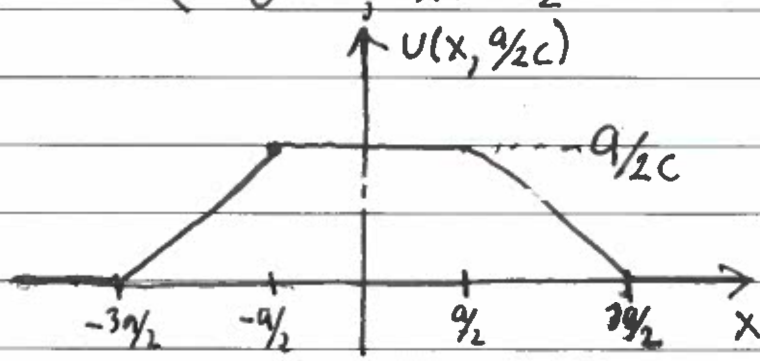
Where $A=(x-ct,x+ct)\cap(-a,a)$. Therefore,

$$u(x,t)=\frac{1}{2c}\left|(x-ct,x+ct)\cap(-a,a)\right|.$$

(c) $t=a/2c$:

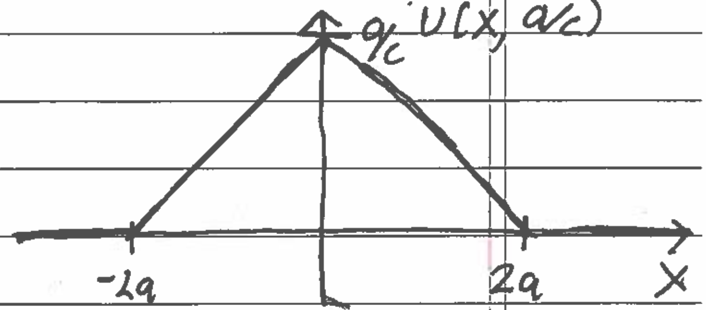
$$u(x,a/2c)=\frac{1}{2c}\left|(x-a/2,x+a/2)\cap(-a,a)\right|$$

$$=\frac{1}{2c}\begin{cases}0 & ,\ x<-3a/2\\ x+3a/2 & ,\ -3a/2<x<-a/2\\ a & ,\ -a/2<x<a/2\\ -x+3a/2 & ,\ a/2<x<3a/2\\ 0 & ,\ x>3a/2\end{cases}$$



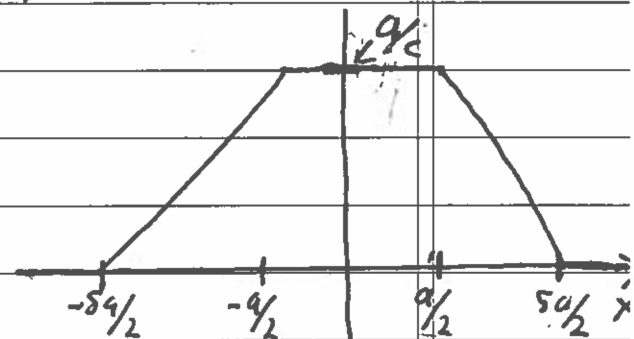
$t = a/c$:

$$u(x, a/c) = \frac{1}{2c}\left|(x-a,\ x+a)\cap(-a,a)\right|$$
$$= \frac{1}{2c}\begin{cases} 0, & x<-2a \\ x+2a, & -2a<x<0 \\ -x+2a, & 0<x<2a \\ 0, & 2a<x \end{cases}$$



$t = 3a/2c$:

$$u(x, 3a/2c) = \frac{1}{2c}\left|(x-3a/2,\ x+3a/2)\cap(-a,a)\right|$$
$$= \frac{1}{2c}\begin{cases} 0, & x<-5a/2 \\ x+5a/2, & -5a/2<x<-a/2 \\ a/c, & -a/2<x<a/2 \\ -x+5a/2, & a/2<x<5a/2 \\ 0, & x>5a/2 \end{cases}$$



The pattern continues like this except the location where the legs of the trapezoid intersect the horizontal axis is located at $x=-a-ct$ and $x=a+ct$ respectively. ■

#4

$$I_x + CV_t + GV = 0, \quad V_x + LI_t + RI = 0$$

Solution:

(a) Differentiating the first equation with respect to t and the second with x we have that

$$I_{xt} + CV_{tt} + GV_t = 0, \quad V_{xx} + LI_{tx} + RI_x = 0$$
$$\Rightarrow CV_{tt} = -I_{xt} - GV_t$$
$$I_{xt} = -\tfrac{1}{L}V_{xx} - \tfrac{R}{L}I_x$$

Now, from the first equation we also have that

$$I_x = -CV_t - GV$$

Therefore,

$$CV_{tt} = \tfrac{1}{L}V_{xx} + \tfrac{R}{L}(-CV_t - GV) - GV_t$$

$$\Rightarrow LCV_{tt} + (RC+LG)V_t + RGV = V_{xx}.$$

A similar derivation works for V!

(b) If we assume $R=G=0$ then we obtain

$$LCV_{tt} = V_{xx}$$

which is the wave equation with speed $c = 1/\sqrt{LC}$.

((c-d) Let $c = 1/\sqrt{LC}$, $a = c^2(LG+RC)$, and $b = c^2RG$ we have

$$U_{tt} + aU_t + bU = c^2U_{xx}$$

Now, if we assume

$$U = e^{-\lambda t}V$$

we have

$$U_t = -\lambda e^{-\lambda t}V + e^{-\lambda t}V_t$$

$$U_{tt} = \lambda^2 e^{-\lambda t}V - 2\lambda e^{-\lambda t}V_t + e^{-\lambda t}V_{tt}$$

$$\Rightarrow \lambda^2 e^{-\lambda t}V - 2\lambda e^{-\lambda t}V_t + e^{-\lambda t}V_{tt} - a\lambda e^{-\lambda t}V + ae^{-\lambda t}V_t$$
$$+ be^{-\lambda t}V = e^{-\lambda t}V_{xx}\cdot c^2$$

$$\Rightarrow V_{tt} + (a-2\lambda)V_t + (\lambda^2 - a\lambda + b)V = c^3V_{xx}$$

If we let $\lambda = a/2$ then

$$V_{tt} + (a^2/4 - a^2/2 + b)V = c^2V_{xx}$$

$$\Rightarrow V_{tt} + (b - a^2/4)V = c^2V_{xx}$$

$$\Rightarrow V_{tt} + KV = c^2V_{xx},$$

where $k = (b - a^2/4)$.

(e) Setting $k=0$ we obtain

$$0 = b - a^2/4$$

$$= c^2RG - c^4\frac{(LG+RC)^2}{4}$$

$$= \frac{RG}{LC} - \frac{L^2G^2 + 2RGLC + R^2C^2}{4L^2C^2}$$

$$\Rightarrow 0 = \frac{4RGLC - L^2G^2 - 2RGL - R^2C^2}{4L^2C^2}$$

$$= -\frac{(LG-RC)^2}{4L^2C^2}$$

$$\Rightarrow LG = RC.$$

(f) If $RC = LG$ then

$$v_{tt} = c^2 v_{xx}$$

$$v(x,0) = f(x)$$

$$\Rightarrow v(x,t) = \tfrac{1}{2} f(x-ct) + \tfrac{1}{2} f(x+ct).$$

Consequently, $u(x,t) = e^{-\lambda t} v(x,t) = \dfrac{e^{-\lambda t}}{2}\left(f(x-ct) + f(x+ct)\right)$.

$\blacksquare$

#5

Solve the following PDE for $x \in \mathbb{R}$ and $t \geq 0$:

$$u_{xx} - 3u_{xt} - 4u_{tt} = 0$$

$$u(x,0) = e^{-x^2}$$

$$u_t(x,0) = 0.$$

Solution:

$$u_{xx} - 3u_{xt} - 4u_{tt} = 0$$

$$\Rightarrow \left(\frac{\partial}{\partial x} - 4\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)u = 0$$

$$\Rightarrow u(x,t) = f(x + 1/4\,t) + g(x-t)$$

$$u_t(x,t) = 1/4\, f'(x + 1/4\,t) - g'(x-t).$$

Therefore,

$$f(x) + g(x) = e^{-x^2}$$

$$1/4\, f'(x) - g'(x) = 0$$

Consequently, $f(x) = 4g(x)$ and thus

$$f(x) + 1/4\, f(x) = e^{-x^2}.$$

and thus

$$f(x) = \tfrac{4}{5} e^{-x^2}.$$

The solution is therefore,

$$u(x,t) = \frac{4}{5} e^{-(x+1/4\,t)^2} + \frac{1}{5} e^{-(x-t)^2}.$$

$\blacksquare$