MTH 352/652
Homework #6

Due Date: March 07, 2025

1. Consider the function $f(x) = x^r$, where $r \in \mathbb{R}$.
   (a) For what values of r is $f(x) \in L^2[0,1]$.
   (b) For what value of r is $f(x) \in L^2[1,\infty)$.
   (c) For what values of r is $f(x) \in L^2[0,\infty)$.
2. Consider the sequence of functions $f_n : \mathbb{R} \mapsto \mathbb{R}$ defined by

$$f_n(x) = \begin{cases} n & \text{if } -\frac{1}{n} < x < 0 \text{ or } 0 < x < \frac{1}{n} \\ 0 & \text{if } x = 0 \text{ or } |x| > \frac{1}{n} \end{cases}.$$

   (a) Explain why $\lim_{n\to\infty} f_n(x) = 0$ pointwise. **Don't go overboard with this. Drawing a picture is fine.**
   (b) Show that f_n does not converge to 0 in L^1 or L^2.

2 3. Consider the sequence of functions $f_n : \mathbb{R} \mapsto \mathbb{R}$ defined by

$$f_n(x) = \begin{cases} n^r & \text{if } -\frac{1}{n} < x < 0 \text{ or } 0 < x < \frac{1}{n} \\ 0 & \text{if } x = 0 \text{ or } |x| > \frac{1}{n} \end{cases},$$

   where $r \in \mathbb{R}$.
   (a) For what values of r does $\lim_{n\to\infty} f_n(x) = 0$ pointwise.
   (b) For what values of r does f_n converge to 0 in L^1.
   (c) For what values of r does f_n converge to 0 in L^2.

2 4. Consider the following infinite series

$$\sum_{n=1}^{\infty}(1-x)x^{n-1}, \quad 0<x<1.$$

(a) Find a simple formula for the partial sums

$$f_m(x)=\sum_{n=1}^{m}(1-x)x^{n-1}$$

by using a geometric series.

(b) Show that the series converges pointwise to some function $f(x)$.

(c) Show that the convergence is not uniform.

(d) Show that the series converges to $f(x)$ in L^2.

5. Consider the following infinite series

$$\sum_{n=1}^{\infty}\left(\frac{n}{1+n^2x^2}-\frac{n-1}{1+(n-1)^2x^2}\right), \quad 0<x<1.$$

(a) Find a simple formula for the partial sums

$$f_m(x)=\sum_{n=1}^{m}\left(\frac{n}{1+n^2x^2}-\frac{n-1}{1+(n-1)^2x^2}\right).$$

(b) Show that the series converges pointwise to some function $f(x)$.

(c) Show that the convergence is not uniform.

(d) Show that the series does not converge in L^2.

2 6. Consider the set of functions

$$\mathcal{A}=\left\{1, \cos\left(\frac{\pi x}{L}\right), \cos\left(\frac{2\pi x}{L}\right), \ldots\right\}.$$

(a) Verify that $\mathcal{A}$ forms an orthogonal system on the interval $[0, L]$, where $L>0$ is a constant.

(b) If

$$f(x)=\sum_{n=0}^{\infty}c_n\cos\left(\frac{n\pi x}{L}\right),$$

in the L^2 sense, what is the formula for c_n? This series is called the Fourier cosine series for f on $[0, L]$.

2 7. Let $f(x)$ be the piecewise function defined by

$$f(x) = \begin{cases} 0, & 0 < x < 1 \\ 1, & 1 < x < 3 \end{cases}.$$

(a) Find the first four nonzero terms of the Fourier cosine series of f.

(b) What is the pointwise limit of the partial sums of this series on $0 \leq x \leq 3$?

(c) Why does this series converge to $f(x)$ in the L^2 sense?

(d) Find the value of the sum

$$1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} + \frac{1}{7} + \frac{1}{8} - \frac{1}{10} - \frac{1}{11} + \ldots.$$

2 8. If c_n are the Fourier coefficients of f with respect to an orthonormal set of functions $\{f_n\}$, show that

$$\left\langle \sum_{i=1}^{N} c_n f_n, f - \sum_{i=1}^{N} c_n f_n \right\rangle = 0.$$

Homework #6

#3

Consider the sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ defined by

$$f_n(x) = \begin{cases} n^r & \text{if } -1/n < x < 0 \text{ or } 0 < x < 1/n, \\ 0 & \text{if } x = 0 \text{ or } |x| > 1/n \end{cases}$$

where $r \in \mathbb{R}$.

(a) For what values of r does $\lim_{n\to\infty} f_n(x) = 0$.
(b) For what values of r does f_n converge to 0 in L^1.
(c) For what values of r does f_n converge to 0 in L^2.

Solution:

(a) For all $x \neq 0$, if $n > 1/|x|$ then $f_n(x) = 0$. Since $f_n(0) = 0$ it follows that

$$\lim_{n\to\infty} f_n(x) = 0.$$

(b) $\int_{-\infty}^{\infty} |f_n(x)|\,dx = 2\int_0^{1/n} n^r\,dx = 2n^{r-1}$. Therefore, if $r < 1$ it follows that $f_n \xrightarrow{L^1} 0$.

(c) $\int_{-\infty}^{\infty} f_n(x)^2\,dx = 2\int_0^{1/n} n^{2r}\,dx = 2n^{2r-1}$. Therefore, if $r < 1/2$ it follows than $f_n \xrightarrow{L^2} 0$.

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#4

Consider the following infinite series

$$\sum_{n=1}^{\infty}(1-x)x^{n-1}, \quad 0<x<1.$$

(a) Find a simple formula for the partial sums

$$f_m(x)=\sum_{n=1}^{m}(1-x)x^{n-1}.$$

(b) Show that the series converges pointwise to some function $f(x)$.

(c) Show that the convergence is not uniform.

(d) Show that the series converges to f in L^2.

Solution:

(a) $f_m(x)=\sum_{n=1}^{m}(1-x)x^{n-1}=(1-x)\sum_{n=1}^{n}x^{n-1}=(1-x)(1-x^n)/1-x=1-x^{n-1}$.

(b) For $x\in(0,1)$, $\lim_{n\to\infty}f_n(x)=1$.

(c) $\lim_{n\to\infty}\max_{0<x<1}|f_n(x)-1|=\lim_{n\to\infty}\max_{0<x<1}x^{n-1}=\lim_{n\to\infty}1=1$. Therefore, $f_n\not\to 1$ uniformly.

(d) $\lim_{n\to\infty}\|f_n-1\|^2=\lim_{n\to\infty}\int_0^1 x^{2n}\,dx=\lim_{n\to\infty}\frac{1}{2n}=0$. Therefore, $f_n\xrightarrow{L^2}0$.

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#6.

Consider the set of functions

$$A=\{1,\cos(\pi x/L),\cos(2\pi x/L),\dots\}.$$

(a) Verify that A forms an orthogonal system on $[0,L]$.

(b) If

$$f(x)=\sum_{n=0}^{\infty}c_n\cos(n\pi x/L),$$

in the L^2 sense, what is the formula for c_n?

#7

Let

$$f(x)=\begin{cases} 0, & 0<x<1 \\ 1, & 1<x<3 \end{cases}.$$

(a) Find the Fourier cosine series of f.

(b) What is the pointwise limit of the partial sums of this series on $0\le x\le 3$.

(c) Why does this series converge to f(x) in the L^2 sense?

(d) Find the value of the sum

$$1+\tfrac{1}{2}-\tfrac{1}{4}-\tfrac{1}{5}+\tfrac{1}{7}+\tfrac{1}{8}-\tfrac{1}{10}-\tfrac{1}{11}+\dots$$

Solution:

(a) $c_0 = \frac{1}{3}\int_0^3 f(x)dx = \frac{1}{3}\int_1^3 dx = \frac{2}{3}.$

$c_n = \frac{2}{3}\int_0^3 f(x)\cos(n\pi x/L)dx = \frac{2}{3}\int_1^3 \cos(n\pi x/3)dx = \frac{2}{n\pi}\sin(n\pi x/3)\Big|_1^3$

Therefore,

$$f(x)\sim \tfrac{2}{3} \; - \tfrac{\sqrt{3}}{\pi}\cos(\pi x/3) - \tfrac{\sqrt{3}}{2\pi}\cos(2\pi x/3) + \tfrac{\sqrt{3}}{3\pi}\cos(\pi x)+\dots)$$

(b) $\lim_{n\to\infty} S_N(x) = \begin{cases} 0, & 0<x<1 \\ 1/2, & x=1 \\ 1, & 1<x<3 \end{cases}$

(c) Since $f\in L^2([0,3])$ it follows that $S_N \xrightarrow{L^2} f.$

(d) Based on the pointwise limit we have that

$$0 = \tfrac{2}{3} - \tfrac{\sqrt{3}}{\pi}\left(1+\tfrac{1}{2}-\tfrac{1}{3}-\tfrac{1}{5}+\tfrac{1}{7}+\tfrac{1}{8}+\dots\right)$$

$$\Rightarrow \tfrac{2\pi}{3\sqrt{3}} = 1+\tfrac{1}{2}-\tfrac{1}{4}-\tfrac{1}{5}+\tfrac{1}{7}+\tfrac{1}{8}+\dots$$

Solution:

(a) For $n \neq 0$ we have

$$\langle 1, \cos(n\pi x/L) \rangle = \int_0^L \cos(n\pi x/L)\,dx$$
$$= L/n\pi \sin(n\pi x/L)\Big|_0^L$$
$$= 0.$$

For $n, m \neq 0$ and $n \neq m$ we have

$$\langle \cos(n\pi x/L), \cos(m\pi x/L) \rangle = \int_0^L \cos(n\pi x/L)\cos(m\pi x/L)\,dx$$
$$= \frac{1}{4}\int_0^L (e^{in\pi x/L} + e^{-in\pi x/L})(e^{im x/L} + e^{-im\pi x/L})$$
$$= \frac{1}{4}\int_0^L (e^{i\pi(n+m)x/L} + e^{-i\pi(n+m)x/L} + e^{i\pi(n-m)x/L}$$
$$+ e^{-i\pi(n-m)x/L})\,dx$$
$$= \frac{1}{2}\int_0^L \cos((n+m)\pi x/L)\,dx + \frac{1}{2}\int_0^L \cos((n-m)\pi x/L)$$
$$= \frac{1}{2}\frac{L}{(n+m)\pi}\sin((n+m)\pi x/L)\Big|_0^L + \frac{1}{2}\frac{L}{(n-m)\pi}\sin((n-m)\pi x/L)$$
$$= 0.$$

Therefore, $\{1, \cos(n\pi x/L)\}$ is an orthogonal system.

(b) If

$$f(x) = \sum_{n=0}^{\infty} c_n \cos(n\pi x/L)$$

then

$$\langle 1, f(x) \rangle = \langle 1, c_0 \rangle = \int_0^L c_0\,dx = L c_0$$
$$\Rightarrow c_0 = \frac{1}{L}\int_0^L f(x)\,dx.$$

We also have that

$$\langle \cos(n\pi x/L), f(x) \rangle = \langle \cos(n\pi x/L), c_n \cos(n\pi x/L) \rangle$$
$$= c_n \int_0^L \cos^2(n\pi x/L)$$
$$= c_n \int_0^L \frac{1}{2}(1 + \cos(2n\pi x/L))$$
$$= \frac{c_n L}{2}$$

$$\Rightarrow c_n = \frac{2}{L}\int_0^L f(x)\cos(2n\pi x/L)\,dx.$$

$\blacksquare$

#8

If c_n are the Fourier coefficients of f with respect to an orthonormal set of functions $\{f_n\}$ show that

$$\langle \sum_{n=1}^{N} c_n f_n, f - \sum_{n=1}^{N} c_n f_n \rangle = 0.$$

Solution:

Assuming $f \sim \sum_{n=1}^{\infty} c_n f_n$ it follows that

$$\langle f, f_n \rangle = c_n \langle f_n, f_n \rangle = c_n.$$

Therefore,

$$\langle \sum_{n=1}^{N} c_n f_n, f - \sum_{n=1}^{N} c_n f_n \rangle = \sum_{n=1}^{N} \langle c_n f_n, f \rangle - \sum_{n=1}^{N} \sum_{m=1}^{N} c_n c_m \langle f_n, f_m \rangle$$

$$= \sum_{n=1}^{N} c_n \langle f_n, f \rangle - \sum_{n=1}^{N} c_n^2$$

$$= \sum_{n=1}^{N} c_n^2 - \sum_{n=1}^{N} c_n^2$$

$$= 0.$$

$\blacksquare$