MTH 352/652
Homework #8

Due Date: March 28, 2025

1. For $f, g \in L^2[a, b]$, prove the Cauchy-Schwarz inequality:

$$|\langle f, g\rangle| \leq \|f\|\,\|g\|.$$

Hint: Define the polynomial $P(t) = \langle f + tg, f + tg\rangle$ for $t \in \mathbb{R}$. Show that $P(t) \geq 0$ which implies that the discriminant of $P(t)$ is negative or zero.

2. For $f, g \in L^2[a, b]$, prove the triangle inequality

$$\|f + g\| \leq \|f\| + \|g\|.$$

Hint: Expand the quantity $\|f + g\|^2$ and apply the Cauchy-Schwarz inequality to the middle term.

3. Consider the function $f(x) = x^2$ on $[-\pi, \pi]$.

(a) Find the Fourier series

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx).$$

(b) Does this Fourier series converge pointwise to x^2?

(c) Does the Fourier series converge in the mean square sense, i.e. L^2 norm, to x^2?

(d) Use the Fourier series to show that

$$\frac{\pi^2}{12} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \ldots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}.$$

4. For $L > 0$, let $V = \{f \in L^2([a, b]) : f'(0) = f'(L) = 0\}$.

(a) Prove that the operator $\mathcal{L}f = \frac{d^2 f}{dx^2}$ is self adjoint on V.

(b) Find the eigenvalues and corresponding eigenfunctions of L on V.

5. Let $\langle \cdot, \cdot \rangle$ denote the complex inner product defined by

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)\overline{g(x)}dx$$

and $V = \{f : \mathbb{R} \mapsto \mathbb{C} : f(x + 2\pi) = f(x)$ and $\langle f, f\rangle < \infty\}$. Show that the operator $\mathcal{L}f = i\frac{df}{dx}$ is self adjoint on V with respect to this inner product.

6. Consider the following initial-boundary value problem on the domain $[0, 2\pi]$:

$$\begin{aligned} u_t &= u_{xx}, \\ u(0,t) &= u(2\pi, t) = 0, \\ u(x,0) &= \sin(x) - \sin(3x) + \sin(5x). \end{aligned}$$

(a) Solve this initial-boundary value problem.

(b) Using software such as Matlab, Mathematica, etc sketch the solution for $t = 0$, $t = .1$, $t = .25$, $t = .5$, and $t = 1$.

(c) For $t = 1$ why does the plot of $u(x,t)$ have the same shape as $\sin(x)$?

7. Solve the following initial-boundary value problem on the the domain $[0, 2\pi]$:

$$\begin{aligned} u_t &= u_{xx}, \\ u_x(0,t) &= u_x(2\pi, t) = 0, \\ u(x,0) &= (x - \pi)^2. \end{aligned}$$

8. Consider the following initial-boundary value problem on the domain $[-\pi, \pi]$:

$$\begin{aligned} u_t &= u_{xx}, \\ u(-\pi,t) &= u(\pi, t), \\ u_x(-\pi,t) &= u_x(\pi, t), \\ u(x,0) &= \begin{cases} 0 & -\pi \le x \le -\frac{\pi}{2}, \\ 1 - |x| & -\frac{\pi}{2} < x < \frac{\pi}{2}, \\ 0 & \frac{\pi}{2} \le x \le \pi. \end{cases}. \end{aligned}$$

(a) What do the boundary conditions model for this problem?

(b) Using separation of variables, solve this initial-value problem.

Homework #8

#1

For $f,g \in L^2([a,b])$, prove the Cauchy-Schwarz inequality:

$$|\langle f,g\rangle| \leq \|f\|\cdot\|g\|.$$

proof:

Let $P(t) = \langle f+tg, f+tg\rangle = \|f+tg\|^2 \geq 0$. Expanding, we have that

$$P(t) = \|f\|^2 + 2t\langle f,g\rangle + t^2\|g\|^2.$$

The roots of $P(t)$ satisfy

$$r_\pm = \frac{-2\langle f,g\rangle \pm \sqrt{4\langle f,g\rangle^2 - 4\|f\|^2\|g\|^2}}{2}.$$

Since $P(t) \geq 0$ it follows that the roots must be complex or repeated and thus

$$4\langle f,g\rangle^2 - 4\|f\|^2\|g\|^2 \leq 0$$

$$\Rightarrow \langle f,g\rangle^2 \leq \|f\|^2\|g\|^2$$

$$\Leftrightarrow |\langle f,g\rangle| \leq \|f\|\cdot\|g\|.$$

$\blacksquare$

#2

For $f,g \in L^2([a,b])$, prove the triangle inequality

$$\|f+g\| \leq \|f\| + \|g\|.$$

proof:

$$\begin{aligned}\|f+g\|^2 &= \langle f+g, f+g\rangle \\ &= \|f\|^2 + 2\langle f,g\rangle + \|g\|^2 \\ &\leq \|f\|^2 + 2|\langle f,g\rangle| + \|g\|^2 \\ &\leq \|f\|^2 + 2\|f\|\cdot\|g\| + \|g\|^2 \\ &\leq (\|f\| + \|g\|)^2\end{aligned}$$

Therefore,

$$\|f+g\| \leq \|f\| + \|g\|.$$

$\blacksquare$

#3.

Consider the function $f(x)=x^2$ on $[-\pi,\pi]$.

(a) Find the Fourier series

$$f(x)\sim a_0+\sum_{n=1}^{\infty}a_n\cos(nx)+\sum_{n=1}^{\infty}b_n\sin(nx)$$

(b) Does the series converge pointwise to x^2?

(c) Does the series converge in the L^2 norm to x^2?

(d) Use the Fourier series to show that

$$\frac{\pi^2}{12}=1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\cdots=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}.$$

Solution:

Taking inner products we have

$$\langle 1,f\rangle=\langle 1,a_0\rangle=\int_{-\pi}^{\pi}a_0\,dx=2\pi a_0$$

$$\Rightarrow a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}x^2dx=\frac{1}{\pi}\int_0^{\pi}x^2dx=\pi^2/3.$$

$$\langle\cos(mx),f\rangle=\langle\cos(mx),a_m\cos(mx)\rangle$$

$$=a_m\int_{-\pi}^{\pi}\cos^2(mx)dx$$

$$=a_m\pi$$

$$\Rightarrow a_m=\frac{1}{\pi}\int_{-\pi}^{\pi}x^2\cos(mx)dx$$

$$=\frac{2}{\pi}\int_0^{\pi}x^2\cos(mx)dx$$

$$=\frac{2}{\pi}\left(\cancel{\frac{x^2\sin(mx)}{m}\Big|_0^{\pi}}^{\,0}-\int_0^{\pi}\frac{2x\sin(mx)\,dx}{m}\right)$$

$$=\frac{2}{\pi}\left(\frac{2x\cos(mx)}{m^2}\Big|_0^{\pi}-\int_0^{\pi}\cancel{\frac{2\cos(mx)}{m^2}}^{\,0}dx\right)$$

$$=\frac{4}{m^2}(-1)^m$$

$$\langle\sin(mx),f\rangle=b_m\pi$$

$$\Rightarrow b_m=\frac{1}{\pi}\int_{-\pi}^{\pi}x^2\sin(mx)dx=0$$

Therefore,

$$x^2 \sim \pi^2/3 + 4\sum_{n=1}^{\infty} \frac{(-1)^m}{m^2}\cos(mx)$$

(b) yes

(c) yes

(d) Evaluating at 0, it follows that

$$0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^m}{m^2}$$

$$\Rightarrow \frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{m+1}}{m^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \dots$$

$\blacksquare$

#4

For $L>0$, let $V = \{f \in L^2([0,L]) : f'(0) = f'(L) = 0\}$.

(a) Prove that the operator $\mathcal{L}f = \frac{d^2f}{dx^2}$ is self adjoint on V

(b) Find the eigenvalues and eigenfunctions on V.

Solution:

(a)
$$\begin{aligned}
\langle \mathcal{L}f, g\rangle &= \int_0^L \frac{d^2f}{dx^2} g\,dx \\
&= \cancel{\frac{df}{dx}g\Big|_0^L} - \int_0^L \frac{df}{dx}\frac{dg}{dx}dx \\
&= \cancel{-f\frac{dg}{dx}\Big|_0^L} + \int_0^L f\frac{d^2g}{dx^2}dx \\
&= \langle f, \mathcal{L}g\rangle.
\end{aligned}$$

Therefore, $\mathcal{L}$ is self adjoint.

(b) Let λ and f satisfy

$$\frac{d^2f}{dx^2} = \lambda f$$

$$\Rightarrow f(x) = A\cos(\sqrt{-\lambda}x) + B\sin(\sqrt{-\lambda}x).$$

$f'(0) = 0 \Rightarrow B = 0$

$f'(L) = 0 \Rightarrow \sqrt{-\lambda}L = n\pi \Rightarrow \lambda = -\frac{n^2\pi^2}{L^2}$, $n = 0, 1, 2, \dots$

Therefore, the eigenvalues are

$$\lambda_n = -\frac{n^2\pi^2}{L^2},\ n \in \{0,1,2,\dots\}$$

with coorresponding eigenfunctions

$$f_n(x) = \cos\left(\frac{n\pi x}{L}\right),\ n \in \{0,1,2,\dots\}$$

■

#5

Let $\langle\cdot,\cdot\rangle$ denote the complex inner product defined by

$$\langle f,g\rangle = \int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx$$

and $V = \{f:\mathbb{R}\to\mathbb{C} : f(x+2\pi)=f(x) \text{ and } \langle f,f\rangle<\infty\}$. Show that the operator $\mathcal{L}f = i\frac{df}{dx}$ is self adjoint on V

Solution:

$$\begin{aligned}\langle \mathcal{L}f,g\rangle &= \int_{-\pi}^{\pi} i\frac{df}{dx}\overline{g(x)}\,dx \\ &= i f(x)\bar{g}(x)\Big|_{-\pi}^{\pi} - i\int_{-\pi}^{\pi} f(x)\overline{\frac{dg}{dx}}\,dx \\ &= \underbrace{\cancel{i f(\pi)\bar{g}(\pi) - i f(-\pi)\bar{g}(-\pi)}}_{0} + \int_{-\pi}^{\pi} f(x)\,\overline{i\frac{dg}{dx}}\,dx \\ &= \int_{-\pi}^{\pi} f(x)\,\overline{\mathcal{L}g}\,dx \\ &= \langle f,\mathcal{L}g\rangle.\end{aligned}$$

Therefore, $\mathcal{L}$ is self adjoint.

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#6.

Consider the following initial-boundary value problem on $[0,2\pi]$:

$$u_t = u_{xx}$$
$$u(0,t) = u(2\pi,t) = 0$$
$$u(x,0) = \sin(x) - \sin(3x) + \sin(5x)$$

(a) Solve this initial-boundary value problem.

(b) Using software, sketch the solution at $t = 0, .1, .25, .5, 1.$

(c) For $t=1$, why does the plot of $u(x,t)$ have the same shape as $\sin(x)$?

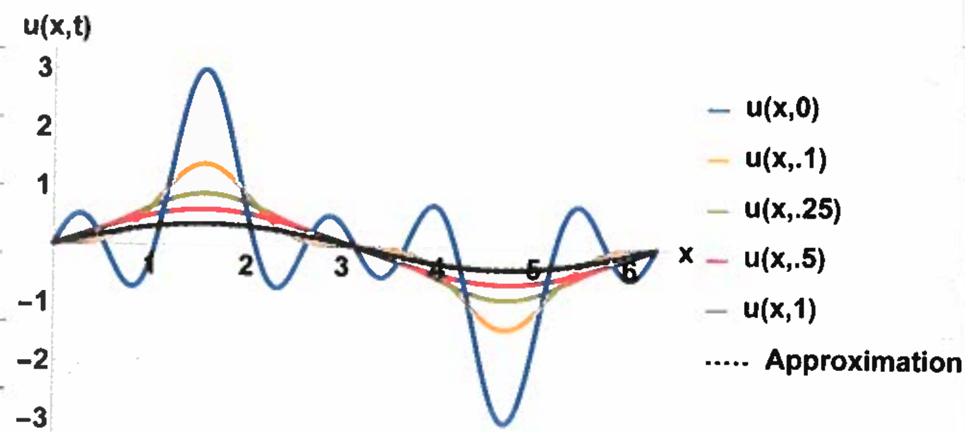
Solution:

(a) Given the intial conditions, we make an ansatz of the form:

$$u(x,t) = e^{-t}\sin(x) - e^{-9t}\sin(3x) + e^{-25t}\sin(5x)$$

which works.

(b).



(c) The dominant term is $e^{-t}\sin(x)$, the other terms have amplitudes $e^{-9} \approx .0001$, and $e^{-25} \approx 1.4 \times 10^{-11}$.

#7

Solve the following initial-boundary value problem on the domain $[-\pi,\pi]$:

$$u_t = u_{xx}$$
$$u_x(0,t) = u_x(2\pi,t) = 0$$
$$u(x,0) = (x-\pi)^2$$

Solution:

Assuming $u(x,t) = X(x)T(t)$ and seperating variables we have that

$$\frac{T'}{T} = \frac{X''}{X} = -\lambda$$
$$\Rightarrow T' = -\lambda T,\ X'' = -\lambda X$$
$$\Rightarrow T = ce^{-\lambda T},\ X = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x).$$

Boundary consitions imply that $B=0$ or $\lambda = 0$. If $\lambda \neq 0$ we obtain

$$\sqrt{\lambda}\cdot 2\pi = n\pi$$
$$\Rightarrow \lambda = n^2/4.$$

Therefore,

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-n^2/4\, t}\cos(nx/2).$$

Consequently,

$$(x-\pi)^2 = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx/2)$$

$$\Rightarrow \int_0^{2\pi}(x-\pi)^2 dx = \int_0^{2\pi} a_0$$
$$\Rightarrow a_0 = \frac{1}{2\pi}\int_0^{2\pi}(x-\pi)^2 dx$$
$$= \frac{1}{6\pi}(x-\pi)^3\Big|_0^{2\pi}$$
$$= \frac{\pi^2}{3}$$

We also have that

$$\int_0^{2\pi}(x-\pi)^2\cos(nx/2)\,dx = \int_0^{2\pi} a_n\cos^2(nx/2)\,dx = \pi a_n.$$

$$\Rightarrow a_n = \frac{1}{\pi}\int_0^{2\pi}\cos(nx/2)(x-\pi)^2\,dx$$

$$= \frac{1}{\pi}\left(\cancelto{0}{\sin(nx/2)(x-\pi)^2\Big|_0^{2\pi}} - \frac{2\cdot 2}{n}\int_0^{2\pi}\sin(nx/2)(x-\pi)\,dx\right)$$

$$= -\frac{4}{n}\int_0^{2\pi}\sin\left(\frac{nx}{2}\right)(x-\pi)\,dx$$

$$= \frac{8}{n^2}\cos\left(\frac{nx}{2}\right)(x-\pi)\Big|_0^{2\pi} - \frac{8}{n^2}\cancelto{0}{\int_0^{2\pi}\cos\left(\frac{nx}{2}\right)dx}$$

$$= \frac{8}{n^2}\cos(n\pi)\pi + \frac{8}{n^2}\pi$$

$$= \frac{8\pi}{n^2}\left((-1)^n+1\right)$$

Therefore,

$$u(x,t) = \frac{\pi^2}{3} + 8\pi\sum_{n=1}^{\infty} e^{-n^2/4\,t}\cos\left(\frac{nx}{2}\right)$$

#8

Consider the following initial value problem on $[-\pi,\pi]$:

$$u_t = u_{xx}$$

$$u(-\pi,t) = u(\pi,t)$$

$$u_x(-\pi,t) = u_x(\pi,t)$$

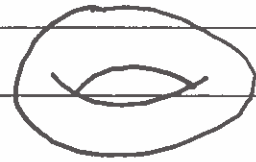
$$u(x,0) = \begin{cases} 0, & |x| \geq \pi/2 \\ 1-|x|, & |x| < \pi/2 \end{cases}$$

(a) What do the boundary conditions model for this problem?

(b) Using seperation of variables solve this initial-value problem.

Solution:

(a) These are periodic boundary conditions which could model the centerline temperature in a torus:



(b). Seperating variables we have that

$$\frac{T'}{T} = \frac{X''}{X} = -\lambda$$

$$\Rightarrow T = ce^{-\lambda t}, \quad X = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x).$$

Boundary conditions imply

$$A\cos(\sqrt{\lambda}\pi) - B\sin(\sqrt{\lambda}\pi) = A\cos(\sqrt{\lambda}\pi) + B\sin(\sqrt{\lambda}\pi)$$

$$A\sqrt{\lambda}\sin(\sqrt{\lambda}\pi) + B\sqrt{\lambda}\cos(\sqrt{\lambda}\pi) = -A\sqrt{\lambda}\sin(\sqrt{\lambda}\pi) + B\sqrt{\lambda}\cos(\sqrt{\lambda}\pi)$$

This has nontrivial solutions if $\lambda = 0$ or $\lambda = n^2$.

Consequently, by linear superposition

$$u(x,t) = a_0 + \sum_{n\geq 1}^{\infty} a_n e^{-n^2 t}\cos(nx) + \sum_{n\geq 1}^{\infty} b_n e^{-n^2 t}\sin(nx).$$

Therefore,

$$u(x,0) = a_0 + \sum_{n\geq 1}^{\infty} a_n\cos(nx) + \sum_{n\geq 1}^{\infty} b_n\sin(nx).$$

Taking inner products we have

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} u(x,0)\sin(nx)\,dx = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}(1-|x|)\sin(nx)\,dx = 0.$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(x,0)\,dx = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}(1-|x|)\,dx = \frac{1}{2\pi}\cdot\pi/2 = 1/4.$$

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(x,0)\cos(nx)\,dx$$

$$= 1/2\pi\int_{-\pi/2}^{\pi/2}(1-|x|)\cos(nx)\,dx$$

$$= 1/\pi\int_0^{\pi/2}(1-x)\cos(nx)\,dx$$

$$= \frac{1}{\pi}\frac{\sin(nx)}{n}\Big|_0^{\pi/2} + \frac{1}{\pi}\int_0^{\pi/2}\frac{\sin(nx)}{n}\,dx$$

$$= \frac{1}{\pi n}\sin\left(\frac{n\pi}{2}\right) - \frac{1}{\pi n^2}\cos(nx)\Big|_0^{\pi/2}$$

$$\Rightarrow a_n = \frac{1}{\pi n}\sin\left(\frac{\pi n}{2}\right) - \frac{1}{\pi n^2}\cos\left(\frac{\pi n}{2}\right) + \frac{1}{\pi n^2}$$

There fore,

$$u(x,t) = 1/4 + \sum_{n=1}^{\infty}\left(\frac{1}{\pi n}\sin\left(\frac{n\pi}{2}\right) - \frac{1}{\pi n^2}\cos\left(\frac{\pi n}{2}\right) + \frac{1}{\pi n^2}\right)e^{-n^2 t}\cos(nx).$$

$\blacksquare$