MTH 352/652
Homework #4

Due Date: February 14, 2025

In all of these problems you can assume that the relevant functions are Schwartz class functions.

1. Verify the following properties of the Fourier transform:
   (a) $\mathcal{F}[u(x)](k) = 2\pi \mathcal{F}^{-1}[u(x)](-k)$.
   (b) $\mathcal{F}[e^{iax}u(x)](k) = \hat{u}(k+a)$.
   (c) $\mathcal{F}[u(x+a)](k) = e^{-iak}\hat{u}(k)$.

2. If f and g are Schwartz class functions, show that

$$(f * g)(x) = (g * f)(x).$$

3. Let $u(x) = e^{-|x|}$.
   (a) Find $\mathcal{F}[u(x)](k)$.
   (b) Find $(u * u)(x)$.
   (c) Find the inverse Fourier transform of the function

$$f(k) = \frac{1}{(1+k^2)^2}.$$

4. Consider the following initial value problem for the heat equation with proportional heat loss:

$$u_t = Du_{xx} - au, \ x \in \mathbb{R}, \ t > 0,$$
$$u(x,0) = e^{-x^2},$$

   where $D > 0$ and $a > 0$ are constants.
   (a) Show that if u is a solution to this PDE then the energy $E(t) = \int_{-\infty}^{\infty} u^2 dx$ is decreasing in time.
   (b) Using this energy, prove that solutions to this PDE are unique.
   (c) Using Fourier transforms, find a formula for the solution to this initial value problem. **Hint:** I am not expecting you to use the convolution theorem to solve this problem. Since $u(x,0)$ is a Gaussian you can explicitly compute all of the Fourier and inverse Fourier transforms.
   (d) Explain what effect of the additional term $-au$ has on the behavior of solutions compared with the heat equation without loss.
   (e) Show that by changing variables $v(x,t) = e^{at}u(x,t)$ that v satisfies the heat equation $v_t = Dv_{xx}$.

5. Consider the following initial value problem for the heat equation with advection:

$$u_t = Du_{xx} - cu_x, \ x \in \mathbb{R}, \ t > 0,$$
$$u(x,0) = e^{-x^2},$$

where $D > 0$ and $c > 0$ are constants.

(a) Show that if u is a solution to this PDE then the energy $E(t) = \int_{-\infty}^{\infty} u^2 dx$ is decreasing in time.

(b) Using this energy, prove that solutions to this PDE are unique.

(c) Using Fourier transforms find a formula for the solution to this initial value problem. **Hint:** I am not expecting you to use the convolution theorem to solve this problem. Since $u(x,0)$ is a Gaussian you can explicitly compute all of the Fourier and inverse Fourier transforms.

(d) Assuming $D = 1$ and $c = 1$, on the same axis sketch $u(x,0)$, $u(x,1)$, and $u(x,2)$ as functions of x. Explain qualitatively the behavior of the solution as time increases.

(e) By changing variables to $\tau = t$ and $X = x - ct$ show that u satisfies $u_\tau = Du_{XX}$.