MTH 352/652
Homework #5

Due Date: February 28, 2025

1. Consider the wave equation for $x \in \mathbb{R}$ and $t > 0$:

$$u_{tt} = c^2 u_{xx}$$

   (a) By taking the Fourier transform of this equation, show that if $u(x,t)$ is a solution then

$$\hat{u}(k,t) = F(k)e^{ikct} + G(k)e^{-ikct},$$

   for some functions F and G.

   (b) If $f(x) = \mathcal{F}^{-1}[F(k)]$ and $g(x) = \mathcal{F}^{-1}[G(k)]$, show that

$$u(x,t) = f(x-ct) + g(x+ct).$$

2. Solve the following PDE on the domain $x \in \mathbb{R}$ and $t \geq 0$:

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \\ u(x,0) &= \frac{1}{1+x^2}, \\ u_t(x,0) &= \sin(x). \end{aligned}$$

3. Let $\Psi(x)$ be the function defined by

$$\Psi(x) = \begin{cases} 1 & |x| < a \\ 0 & |x| \geq 0 \end{cases},$$

for some constant $a > 0$. Suppose $u(t,x)$ is a solution to the following PDE

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \\ u(x,0) &= 0, \\ u_t(x,0) &= \Psi(x). \end{aligned}$$

   (a) Sketch a graph of $\Psi(x)$.

   (b) Show that the solution to this PDE is given by

$$u(x,t) = \frac{1}{2c}|(x-ct, x+ct) \cap (-a,a)|,$$

   where here $|\cdot|$ denotes the length of an interval on the real line.

   (c) Sketch the solution at times $t = a/2c$, $t = a/c$, $t = 3a/2c$, $t = 2a/c$, and $t = 5a/c$.

4. In a long transmission line the voltage $V(x,t)$ and current $I(x,t)$ satisfy the transmission line equations:

$$I_x + CV_t + GV = 0 \text{ and } V_x + LI_t + RI = 0,$$

where $C > 0$ is the capacitance, $G > 0$ is the leakage, $R > 0$ is the resistance, and $L > 0$ is the inductance.

(a) Show that both V and I satisfy the telegraph equation

$$LCu_{tt} + (RC + LG)u_t + RGu = u_{xx}. \tag{1}$$

(b) If for the moment we assume that $R = G = 0$ show that Equation (1) become the wave equation. What is the speed? Why is this case unrealistic?

(c) If we let $c = /\sqrt{LC}$, $a = c^2(LG + RC)$ and $b = c^2 RG$ show that Equation (1) becomes

$$u_{tt} + au_t + bu = c^2 u_{xx}.$$

(d) By making a substitution of the form $u(x,t) = e^{-\lambda t}v(x,t)$ and by appropriately picking λ in terms of a, b, and c, show that Equation (4c) becomes

$$v_{tt} + kv = c^2 v_{xx},$$

where k is a constant that depends on a, b, and c.

(e) Show that $k = 0$ only when $RC = LG$.

(f) In the case when $RC = LG$ with the initial conditions $v(x,0) = f(x)$ and $v_t(x,0) = 0$, show that a solution to the telegrapher's equation is given by

$$u(t,x) = \frac{e^{-\lambda t}}{2}\left(f(x - ct) + f(x + ct)\right).$$

What does this solution mean in practical terms? In particular, why is it useful for sending signals?

Remark This process of tuning the electrical parameters so the $RC = LG$ called Pupinizing the cable after one of its discoverers Michael Pupin.

5. Solve the following PDE for $x \in \mathbb{R}$ and $t \geq 0$:

$$\begin{aligned} u_{xx} - 3u_{xt} - 4u_{tt} &= 0, \\ u(x,0) &= e^{-x^2}, \\ u_t(x,0) &= e^x. \end{aligned}$$

Hint: Factor the differential operator like we did in class.