MTH 352/652
Homework #6

Due Date: March 07, 2025

1. Consider the function $f(x) = x^r$, where $r \in \mathbb{R}$.
   (a) For what values of r is $f(x) \in L^2[0,1]$.
   (b) For what value of r is $f(x) \in L^2[1,\infty)$.
   (c) For what values of r is $f(x) \in L^2[0,\infty)$.

2. Consider the sequence of functions $f_n : \mathbb{R} \mapsto \mathbb{R}$ defined by

$$f_n(x) = \begin{cases} n & \text{if } -\frac{1}{n} < x < 0 \text{ or } 0 < x < \frac{1}{n} \\ 0 & \text{if } x = 0 \text{ or } |x| > \frac{1}{n} \end{cases}.$$

   (a) Explain why $\lim_{n\to\infty} f_n(x) = 0$ pointwise. **Don't go overboard with this. Drawing a picture is fine**.
   (b) Show that f_n does not converge to 0 in L^1 or L^2.

3. Consider the sequence of functions $f_n : \mathbb{R} \mapsto \mathbb{R}$ defined by

$$f_n(x) = \begin{cases} n^r & \text{if } -\frac{1}{n} < x < 0 \text{ or } 0 < x < \frac{1}{n} \\ 0 & \text{if } x = 0 \text{ or } |x| > \frac{1}{n} \end{cases},$$

   where $r \in \mathbb{R}$.
   (a) For what values of r does $\lim_{n\to\infty} f_n(x) = 0$ pointwise.
   (b) For what values of r does f_n converge to 0 in L^1.
   (c) For what values of r does f_n converge to 0 in L^2.

4. Consider the following infinite series

$$\sum_{n=1}^{\infty}(1-x)x^{n-1},\ 0<x<1.$$

   (a) Find a simple formula for the partial sums

$$f_m(x)=\sum_{n=1}^{m}(1-x)x^{n-1}$$

   by using a geometric series.

   (b) Show that the series converges pointwise to some function $f(x)$.

   (c) Show that the convergence is not uniform.

   (d) Show that the series converges to $f(x)$ in L^2.

5. Consider the following infinite series

$$\sum_{n=1}^{\infty}\left(\frac{n}{1+n^2x^2}-\frac{n-1}{1+(n-1)^2x^2}\right),\ 0<x<1.$$

   (a) Find a simple formula for the partial sums

$$f_m(x)=\sum_{n=1}^{m}\left(\frac{n}{1+n^2x^2}-\frac{n-1}{1+(n-1)^2x^2}\right).$$

   (b) Show that the series converges pointwise to some function $f(x)$.

   (c) Show that the convergence is not uniform.

   (d) Show that the series does not converge in L^2.

6. Consider the set of functions

$$\mathcal{A}=\left\{1,\cos\left(\frac{\pi x}{L}\right),\cos\left(\frac{2\pi x}{L}\right),\dots\right\}.$$

   (a) Verify that $\mathcal{A}$ forms an orthogonal system on the interval $[0,L]$, where $L>0$ is a constant.

   (b) If

$$f(x)=\sum_{n=0}^{\infty}c_n\cos\left(\frac{n\pi x}{L}\right),$$

   in the L^2 sense, what is the formula for c_n? This series is called the Fourier cosine series for f on $[0,L]$.

7. Let $f(x)$ be the piecewise function defined by

$$f(x) = \begin{cases} 0, & 0 < x < 1 \\ 1, & 1 < x < 3 \end{cases}.$$

(a) Find the first four nonzero terms of the Fourier cosine series of f.

(b) What is the pointwise limit of the partial sums of this series on $0 \leq x \leq 3$?

(c) Why does this series converge to $f(x)$ in the L^2 sense?

(d) Find the value of the sum

$$1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} + \frac{1}{7} + \frac{1}{8} - \frac{1}{10} - \frac{1}{11} + \ldots.$$

8. If c_n are the Fourier coefficients of f with respect to an orthonormal set of functions $\{f_n\}$, show that

$$\left\langle \sum_{i=1}^{N} c_n f_n, f - \sum_{i=1}^{N} c_n f_n \right\rangle = 0.$$