MTH 352/652
Homework #8

Due Date: March 28, 2025

1. For $f, g \in L^2[a,b]$, prove the Cauchy-Schwarz inequality:

$$|\langle f, g\rangle| \leq \|f\| \, \|g\|.$$

Hint: Define the polynomial $P(t) = \langle f + tg, f + tg\rangle$ for $t \in \mathbb{R}$. Show that $P(t) \geq 0$ which implies that the discriminant of $P(t)$ is negative or zero.

2. For $f, g \in L^2[a,b]$, prove the triangle inequality

$$\|f + g\| \leq \|f\| + \|g\|.$$

Hint: Expand the quantity $\|f + g\|^2$ and apply the Cauchy-Schwarz inequality to the middle term.

3. Consider the function $f(x) = x^2$ on $[-\pi, \pi]$.

   (a) Find the Fourier series

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx).$$

   (b) Does this Fourier series converge pointwise to x^2?

   (c) Does the Fourier series converge in the mean square sense, i.e. L^2 norm, to x^2?

   (d) Use the Fourier series to show that

$$\frac{\pi^2}{12} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \ldots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}.$$

4. For $L > 0$, let $V = \{f \in L^2([a,b]) : f'(0) = f'(L) = 0\}$.

   (a) Prove that the operator $\mathcal{L}f = \frac{d^2 f}{dx^2}$ is self adjoint on V.

   (b) Find the eigenvalues and corresponding eigenfunctions of L on V.

5. Let $\langle \cdot, \cdot \rangle$ denote the complex inner product defined by

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)\overline{g(x)} dx$$

and $V = \{f : \mathbb{R} \mapsto \mathbb{C} : f(x + 2\pi) = f(x) \text{ and } \langle f, f\rangle < \infty\}$. Show that the operator $\mathcal{L}f = i\frac{df}{dx}$ is self adjoint on V with respect to this inner product.

6. Consider the following initial-boundary value problem on the domain $[0, 2\pi]$:

$$
\begin{aligned}
&u_t = u_{xx}, \\
&u(0,t) = u(2\pi, t) = 0, \\
&u(x,0) = \sin(x) - \sin(3x) + \sin(5x).
\end{aligned}
$$

   (a) Solve this initial-boundary value problem.

   (b) Using software such as Matlab, Mathematica, etc sketch the solution for $t = 0$, $t = .1$, $t = .25$, $t = .5$, and $t = 1$.

   (c) For $t = 1$ why does the plot of $u(x,t)$ have the same shape as $\sin(x)$?

7. Solve the following initial-boundary value problem on the the domain $[0, 2\pi]$:

$$
\begin{aligned}
u_t &= u_{xx}, \\
u_x(0,t) &= u_x(2\pi, t) = 0, \\
u(x,0) &= (x-\pi)^2.
\end{aligned}
$$

8. Consider the following initial-boundary value problem on the domain $[-\pi, \pi]$:

$$
\begin{aligned}
&u_t = u_{xx}, \\
&u(-\pi,t) = u(\pi, t), \\
&u_x(-\pi,t) = u_x(\pi, t), \\
&u(x,0) = \begin{cases} 0 & -\pi \le x \le -\frac{\pi}{2}, \\ 1 - |x| & -\frac{\pi}{2} < x < \frac{\pi}{2}, \\ 0 & \frac{\pi}{2} \le x \le \pi. \end{cases} .
\end{aligned}
$$

   (a) What do the boundary conditions model for this problem?

   (b) Using separation of variables, solve this initial-value problem.